SOME PROBLEMS ON FLUID MOTION, WITH SPECIAL REFERENCE TO THE FLOW OF COMPRESSIBLE FLUIDS, AND WITH ADDITIONAL PAPERS.

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P R E F A C E

This thesis is mainly concerned with problems relating to the two-dimensional flow of a compressible fluid at high speed. Chapter I gives a brief resume of certain standard results which are frequently used later, while Chapter II deals with the so-called hodograph method of solving problems on compressible flow. This method has, in some respects, been treated in fair detail because an adequate account is only to be found in journals which are comparatively inaccessible. These two chapters contain nothing original and are included in order to make the work reasonably self-contained. In Chapter III a known result, concerning the convergence of certain series associated with the hodograph method, is extended and, at the same time, approximate solutions of the hodograph equations are obtained. These approximations are then applied in Chapter IV to the solution of a particular problem. Chapter V is concerned with an entirely different aspect of high speed flow, viz., to estimate the increase of drag on a body (moving through a compressible fluid) due to the presence of a shock wave. This chapter ends the work on the flow of compressible fluids.

In Chapter VI a problem on the slow motion of a viscous fluid is discussed.

The first of the additional papers, "On the Fluxgate Principle", attempts to place on a fairly rigorous basis a certain result in electromagnetism which is now being widely used in the design of aircraft compasses and accurate magnetometers and which has so far had very little theoretical justification. I was led to consider this problem during the war years, while working in the Navigation Section of the Royal Aircraft Establishment, Farnborough, and completed the work after returning to Glasgow.

The paper on Dirac's equation contains an enquiry into the compatibility of various suggested methods of extending this equation to General Relativity.

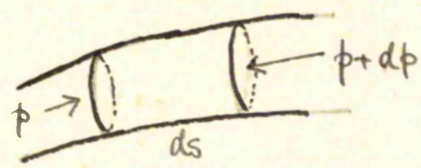
The remaining papers "On the Radial Error in a Gaussian Elliptical Scatter" and "Some Properties of the Curve of Constant Bearing" are largely exercises on Bessel Functions and Spherical Trigonometry, respectively.

Finally, I wish to thank Professor R.O.Street of the Royal Technical College, Glasgow, for supervising my research and for helping me in many ways while I was a member of his staff.

I. Some standard results of the theory of the flow of a compressible fluid in one and two dimensions

We begin by collecting here, for reference, the more important formulae for the one-dimensional flow of a frictionless compressible fluid which are derived from Bernoulli's Theorem and which will be required later in our work.

Bernoulli's Theorem for steady stream-line motion



Consider a small section of a stream tube, of length ds and cross-sectional area S. Then, since the motion is steady, it follows from the Second Law of Motion that

$$\rho S ds \times q \frac{dq}{ds} = pS - (p+dp)S$$

where q is the velocity of the element, ρ is its density and p and $p+dp$ are the pressures acting at the ends,

i.e.

$$q \frac{dq}{ds} + \frac{1}{\rho}\frac{dp}{ds} = 0 .$$

Integrating along the stream tube, we have

$$\tfrac{1}{2}q^2 + \int \frac{dp}{\rho} = \text{const.}, \tag{1.1}$$

which is Bernoulli's Theorem. In general, the constant of integration varies from one streamline to another.

The velocity of sound c at any point in the fluid is given by

$$c^2 = \frac{dp}{d\rho}$$

and, if adiabatic conditions prevail, i.e.

$$p\rho^{-\gamma} = \text{constant},$$

where γ is the ratio of the specific heats of the fluid at constant pressure and constant volume, then

$$c^2 = \gamma p/\rho \propto \rho^{\gamma-1}.$$

If zero subscripts refer to the state of the fluid at rest, i.e. at the stagnation point when we are contemplating the flow of the fluid past a solid body, then (1.1) gives

$$\frac{p}{p_0} = \left\{1 - \frac{\gamma-1}{2}\left(\frac{q}{c_0}\right)^2\right\}^{\frac{\gamma}{\gamma-1}}, \tag{1.2}$$

$$\frac{\rho}{\rho_0} = \left\{1 - \frac{\gamma-1}{2}\left(\frac{q}{c_0}\right)^2\right\}^{\frac{1}{\gamma-1}} \tag{1.3}$$

and

$$\left(\frac{c}{c_0}\right)^2 = 1 - \frac{\gamma-1}{2}\left(\frac{q}{c_0}\right)^2. \tag{1.4}$$

It is often convenient to write (1.4) in the form

$$q^2 = 2\beta(c_0^2 - c^2), \tag{1.5}$$

where $\beta = 1/(\gamma-1) = 2.5$ for air.

It follows immediately from (1.5) that there is a maximum velocity q_m attained by the gas when $c = 0$;

when $c=0$, $\rho=0$ and $p=0$. Thus

$$q_m^2 = 2\beta c_0^2. \tag{1.6}$$

The so-called critical speed of the fluid q_s, which is attained when the local speed of the fluid and the local speed of sound are equal, is of great importance. By (1.5) we see that

$$(2\beta+1) q_s^2 = 2\beta c_0^2 = q_m^2. \tag{1.7}$$

At this point, it is useful to introduce[1] the non-dimensional parameter τ, given by

$$\tau = q^2/q_m^2, \qquad (0 \leqq \tau \leqq 1).$$

For an incompressible fluid τ is zero at every point of the field of flow. Formulae (1.2), (1.3) and (1.4) may now be written

$$p = p_0(1-\tau)^{\beta+1}, \tag{1.8}$$

$$\rho = \rho_0(1-\tau)^{\beta}, \tag{1.9}$$

$$c = c_0(1-\tau)^{1/2}. \tag{1.10}$$

τ_s the value of τ corresponding to the critical speed q_s is given by

$$\tau_s = \frac{q_s^2}{q_m^2} = \frac{1}{2\beta+1} = \frac{1}{6} \text{ for air.}$$

Thus the subsonic region of the field of flow is characterised by $0 \leqq \tau < \frac{1}{2\beta+1}$ and the supersonic region by $\frac{1}{2\beta+1} < \tau \leqq 1$.

It is useful also to obtain the relationship between τ

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1. Following Caius Jacob: Bull. Scientifique de l'Ecole Polytechnique de Timisoara, t.7(1937) pp. 47-59.

and the Mach number M, which is defined by

$$M = q/c.$$

By (1.5) and (1.10),

$$M^2 = 2\beta\left(\frac{c_0^2}{c^2} - 1\right) = 2\beta\left(\frac{1}{1-\tau} - 1\right)$$

$$\text{i.e.} \quad M^2 = \frac{2\beta\tau}{1-\tau} \tag{1.11}$$

We pass on now to consider the two-dimensional irrotational motion of a compressible fluid.

Two-Dimensional Irrotational Motion of a compressible Fluid

Let u, v denote the x, y components respectively of the velocity q. Then the absence of vorticity implies that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{1.12}$$

while the continuity condition gives

$$\frac{\partial}{\partial x}\left(\frac{\rho u}{\rho_0}\right) + \frac{\partial}{\partial y}\left(\frac{\rho v}{\rho_0}\right) = 0. \tag{1.13}$$

It follows from (1.12) that $u\,dx + v\,dy$ is the complete differential $d\varphi$ of some function φ of x and y,

i.e. $$u\,dx + v\,dy = d\varphi$$

and $$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}; \tag{1.14}$$

φ is called the velocity potential.

Similarly from (1.13)

$$-\frac{\rho v}{\rho_0}\,dx + \frac{\rho u}{\rho_0}\,dy = d\psi,$$

where ψ is another function of x and y, called the stream function, and

$$u = \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial y}, \quad v = -\frac{\rho_0}{\rho}\frac{\partial \psi}{\partial x}. \tag{1.15}$$

From (1.14) and (1.15) we conclude that

$$\frac{\partial \varphi}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\rho_0}{\rho}\frac{\partial \psi}{\partial x}. \tag{1.16}$$

In the case of an incompressible fluid $\rho = \rho_0$ everywhere and equations (1.16) are simply the Cauchy Riemann Equations which express the fact that $w \equiv \varphi + i\psi$ is a holomorphic function of $z \equiv x + iy$. Unfortunately, in the general case, w is not a holomorphic function of z, and so the powerful methods of the theory of the complex variable are not at our disposal for the exact solution of problems on compressible flow.

We now proceed to derive the basic partial differential equation which is satisfied by φ and ψ.

Making use of the vector calculus, we may write the equation of continuity (1.13) in the form

$$\rho \operatorname{div} \underset{\sim}{q} + \underset{\sim}{q} \cdot \operatorname{grad} \rho = 0,$$

or, by (1.14)

$$\nabla^2 \varphi + \frac{1}{\rho} \operatorname{grad} \varphi \cdot \operatorname{grad} \rho = 0. \tag{1.17}$$

Now, from Bernoulli's Theorem (1.1),

$$\frac{1}{2} \operatorname{grad} \underset{\sim}{q}^2 + \frac{1}{\rho}\frac{dp}{d\rho} \operatorname{grad} \rho = 0$$

or

$$\frac{1}{\rho} \operatorname{grad} \rho = -\frac{1}{2c^2} \operatorname{grad} \underset{\sim}{q}^2 . \qquad (1.18)$$

Hence, by (1.17) and (1.18)

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{1}{2c^2}\left[\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial y}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\}\right] = 0$$

i.e.

$$\frac{\partial^2\varphi}{\partial x^2}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial^2\varphi}{\partial y^2}\left(1 - \frac{v^2}{c^2}\right) - \frac{2uv}{c^2}\frac{\partial^2\varphi}{\partial x \partial y} = 0. \qquad (1.19)$$

Similarly, by expressing u, v in terms of ψ instead of φ we arrive at the equation

$$\frac{\partial^2\psi}{\partial x^2}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial^2\psi}{\partial y^2}\left(1 - \frac{v^2}{c^2}\right) - \frac{2uv}{c^2}\frac{\partial^2\psi}{\partial x \partial y} = 0. \qquad (1.20)$$

In (1.19) and (1.20) u, v and c^2 must be expressed in terms of the derivatives of φ or ψ, and it is evident that both equations are non-linear and exceedingly complicated. It should be observed that φ and ψ do not satisfy the same equation because u and v depend on φ and ψ in different ways.

(1.19) and (1.20) are the basic partial differential equations for the flow of a compressible fluid.

II. Methods of solution of the fundamental partial differential equation of two dimensional compressible flow

The fundamental equation (1.19) or (1.20) is a partial differential equation of the second order and of mixed type. It is of elliptical type when $q^2 \equiv u^2+v^2 < c^2$ (i.e. in the subsonic region), is hyperbolic when $q^2 > c^2$ (i.e. in the supersonic region) and is parabolic when $q = c$. The difficulties entailed in the integration are due partly to the non-linear character of the equation and partly to the change of type at some critical value of the velocity of sound which at first is still unknown and has to be determined in the course of the calculation itself.

We now consider two of the more important methods which have been developed to overcome these difficulties. For a résumé of the other methods (e.g. the Janzen-Rayleigh Iteration Method, G.I. Taylor's electrical method etc.) reference should be made to a paper by Eser[1].

The Linear Perturbation Theory

This theory provides an approximate method of

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1. Eser: Luftfahrtforschung 20(1943), 220

determining the <u>subsonic</u> flow of a fluid past a slender profile and is due to Prandtl[1.] and Glauert[2.] As a very full account can be found in a recent report by Goldstein and Young[3.], we shall give here only a very brief outline of the theory and mention two formulae which we shall require later.

Let the x-axis be taken along the direction of the undisturbed flow and let the velocity at an infinite distance from the profile be U; U may be of the order of the local velocity of sound. We then assume that in the neighbourhood of the profile

(a) the x-component of the velocity, u, differs only slightly from U,

(b) the y-component of the velocity, v, is negligible,

(c) the velocity of sound c may be replaced by its value in the undistrubed stream.

Equation (1.20) for the stream function then reduces to

$$(1-M^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \tag{2.1}$$

where M is the Mach. number of the undisturbed stream. This equation may be written

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$$

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1. Prandtl: Journ. Aeronautical Research Inst., Tokyo Imp. Univ. No. 65 (1930), 14.
2. Glauert: Proc. Roy. Soc. A 118 (1928), 113
3. Goldstein & Young: Reports & Memoranda Aeronautical Research Committee No. 1909 (1943)

where $\xi = x(1-M^2)^{-1/2}$, $\eta = y$.

Hence the compressible field of flow around the profile can be obtained from the incompressible field of flow around a profile whose thickness ratio differs from that of the given profile by a factor $(1-M^2)^{1/2}$. The theory breaks down near any stagnation point on the profile for, at such a point, the difference between u and U is equal to U itself.

If the pressure at some point in the field of flow past a given profile is p and the pressure in the undisturbed stream is p_1, then it can be shown that

$$(p-p_1)_{compressible} = \frac{(p-p_1)_{incompressible}}{\sqrt{(1-M^2)}}, \tag{2.2}$$

where M is the Mach. number of the undisturbed stream.

Further, the theory provides an approximate method of calculating the critical Mach. number of the profile, i.e. the Mach number (of the undisturbed flow) at which the velocity of sound is first attained at some point of the profile. The critical Mach number is usually expressed as a function of q_{max}/U where q_{max} is the maximum velocity on the profile calculated for incompressible flow. The relationship turns out to be

(2.3)

The Hodograph Method

We shall treat this method in considerable detail because a complete account of it is only to be found in memoirs which are comparatively inaccessible. The principle of the method was explained in 1890 by Molenbroeck[1] for a gas for which $\gamma = -1$ and then extended and made applicable to any gas by Tchapliguine[2] in 1904. However, little heed was paid to these papers until Demtschenko[3] drew attention to them in 1932. The account given below follows closely that of Caius Jacob[4], who also applied the method to the theory of gas jets. Use has also been made of a paper by Ringleb[5] in which some exact solutions of the fundamental equations are derived by the method.

The essential feature of the method is that the basic partial differential equation (1.20) defining the flow of a compressible fluid can be transformed into a <u>linear</u> equation by changing the independent variables from x, y to q, θ where q is the resultant

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1. Molenbroeck: Archiv. d. Math.u. Phys. Grunnert-Hoppe II Vol. 9 (1890) 70-157
2. Tchapliguine: Ann. scientifique de l'Univ. de Moscou (1904), 1-21
3. Demtschenko: C.R.194(1932),1218; C.R.(1932),1720
4. Jacob: Bull.Sci. de l'Ecole Polytechnique de Timisoara (1937) 47-59, 224-244
5. Ringleb: Z.A.M.M. 20(1940), 185-198

velocity of the fluid at any point and $\theta \equiv \tan^{-1}\frac{v}{u}$ is the angle which this resultant velocity makes with the x-axis. For the subsequent application of the theory, it is more convenient to take the independent variables as $\tau\ (\equiv q^2/q_m^2)$ and θ rather than q and θ.

From the equations (cf. (1.14) and (1.15)),

$$d\varphi = u\,dx + v\,dy, \qquad d\psi = \frac{\rho}{\rho_0}(-v\,dx + u\,dy),$$

we have

$$dx = \frac{1}{u^2+v^2}\left(u\,d\varphi - \frac{\rho_0}{\rho}v\,d\psi\right) = \frac{1}{q_m\tau^{1/2}}\left(\cos\theta\,d\varphi - \frac{\rho_0}{\rho}\sin\theta\,d\psi\right), \tag{2.4}$$

$$dy = \frac{1}{u^2+v^2}\left(v\,d\varphi + \frac{\rho_0}{\rho}u\,d\psi\right) = \frac{1}{q_m\tau^{1/2}}\left(\sin\theta\,d\varphi + \frac{\rho_0}{\rho}\cos\theta\,d\psi\right).$$

Now, in an obvious notation,

$$d(x,y,\varphi,\psi) = (x,y,\varphi,\psi)_\tau\,d\tau + (x,y,\varphi,\psi)_\theta\,d\theta,$$

whence, by (2.4)

$$x_\tau = \frac{1}{q_m\tau^{1/2}}\left(\cos\theta\,\varphi_\tau - \frac{\rho_0}{\rho}\sin\theta\,\psi_\tau\right),$$

$$x_\theta = \frac{1}{q_m\tau^{1/2}}\left(\cos\theta\,\varphi_\theta - \frac{\rho_0}{\rho}\sin\theta\,\psi_\theta\right), \tag{2.5}$$

$$y_\tau = \frac{1}{q_m\tau^{1/2}}\left(\sin\theta\,\varphi_\tau + \frac{\rho_0}{\rho}\cos\theta\,\psi_\tau\right),$$

$$y_\theta = \frac{1}{q_m\tau^{1/2}}\left(\sin\theta\,\varphi_\theta + \frac{\rho_0}{\rho}\cos\theta\,\psi_\theta\right).$$

The conditions $x_{\tau\theta} = x_{\theta\tau}$, $y_{\tau\theta} = y_{\theta\tau}$ then yield

$$\sin\theta\,\varphi_\tau - \frac{\cos\theta}{2\tau}\varphi_\theta = -\frac{\rho_0}{\rho}\cos\theta\,\psi_\tau + \sin\theta\left[\left(\frac{\rho_0}{\rho}\right)_\tau - \frac{1}{2\tau}\frac{\rho_0}{\rho}\right]\psi_\theta, \tag{2.6}$$

and

$$\cos\theta\,\varphi_\tau + \frac{\sin\theta}{2\tau}\varphi_\theta = \frac{\rho_0}{\rho}\sin\theta\,\psi_\tau + \cos\theta\left[\left(\frac{\rho_0}{\rho}\right)_\tau - \frac{1}{2\tau}\frac{\rho_0}{\rho}\right]\psi_\theta,$$

whence

$$\varphi_\tau = \psi_\theta\left[\left(\frac{\rho_0}{\rho}\right)_\tau - \frac{1}{2\tau}\frac{\rho_0}{\rho}\right] \tag{2.7}$$

and $$\varphi_\theta = 2\tau \frac{\rho_0}{\rho} \psi_\tau . \tag{2.8}$$

By (1.9) these equations may be written

$$\varphi_\tau = \frac{(2\beta+1)\tau - 1}{2\tau(1-\tau)^{\beta+1}} \psi_\theta \tag{2.9}$$

and $$\varphi_\theta = 2\tau(1-\tau)^{-\beta} \psi_\tau . \tag{2.10}$$

Equations (2.7) and (2.8) are the general hodograph equations corresponding to equations (1.16), and are true for all pressure - density relationships; they assume the forms (2.9) and (2.10) when adiabatic conditions are postulated, as will be done in the problems with which we are concerned here.

It follows immediately from (2.9) and (2.10) that

$$\frac{\partial}{\partial\tau}\left[2\tau(1-\tau)^{-\beta} \frac{\partial\psi}{\partial\tau}\right] + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}} \frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{2.11}$$

This equation determines ψ, whence φ can be determined by (2.9) and (2.10).

We now proceed to solve the equation (2.11)

Putting $\psi = T\Theta$ where T, Θ are respectively functions of τ and θ alone, we have

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\frac{4\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau} \frac{1}{T} \frac{d}{d\tau}\left[\tau(1-\tau)^{-\beta}\frac{dT}{d\tau}\right].$$

Arguing in the usual way, we put each side of this equation equal to a constant $-n^2$, and get

$$\Theta = B_n' \sin(n\theta + \mu_n)$$

where B_n' and μ_n are constants, and

$$\tau(1-\tau)\frac{d^2T}{d\tau^2} + \left[(\beta-1)\tau + 1\right]\frac{dT}{d\tau} - \frac{1-(2\beta+1)\tau}{4\tau} n^2 T = 0. \tag{2.12}$$

We next put $T = \tau^{n/2} u$ whence

$$\frac{dT}{d\tau} = \tau^{\frac{n}{2}} \frac{du}{d\tau} + \tfrac{1}{2} n \tau^{\frac{1}{2}n-1} u,$$

$$\frac{d^2T}{d\tau^2} = \tau^{n/2} \frac{d^2u}{d\tau^2} + n\tau^{\frac{n}{2}-1} \frac{du}{d\tau} + \tfrac{1}{2} n \left(\tfrac{1}{2}n - 1\right) \tau^{\frac{1}{2}n-2} u,$$

and (2.12) becomes, on simplification

$$\tau(1-\tau)\frac{d^2u}{d\tau^2} + \left[n+1-\tau(n-\beta+1)\right]\frac{du}{d\tau} + \tfrac{1}{2}n(n+1)\beta u = 0. \qquad (2.13)$$

Comparing this equation with the Hypergeometric Equation

$$\tau(1-\tau)y'' + \left[\gamma - (\alpha+\beta+1)\tau\right]y' - \alpha\beta y = 0,$$

one of whose fundamental solutions is $y = F(\alpha, \beta, \gamma, \tau)$,

we see that a solution of (2.13) is

$$y_n(\tau) = F(a_n, b_n, c_n, \tau)$$

where

$$a_n + b_n = n - \beta,$$

$$a_n b_n = -\tfrac{1}{2} n(n+1)\beta,$$

$$c_n = n+1.$$

Thus, a formal solution of (2.11) is

$$\psi(\theta, \tau) = A + B\theta + \sum_n B_n \left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n} \frac{y_n(\tau)}{y_n(\tau_1)} \sin(n\theta + \mu_n), \qquad (2.14)$$

where A, B, B_n, μ_n are constants, τ_1 is the (constant) value of τ for the main stream (we are contemplating the flow of the fluid past solid bodies) and the summation is taken over values of n not yet specified. The terms $A + B\theta$ are obtained by taking n and T both zero, and there is no loss of generality in taking A equal to zero. The reason for introducing the constant τ_1, in the solution will be apparent later.

Now, by (2.10)

$$\frac{\partial\varphi}{\partial\theta} = \frac{2\tau}{(1-\tau)^{\beta}}\sum_n B_n\left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n}\left[\frac{y_n'(\tau)}{y_n(\tau_1)} + \frac{n}{2\tau}\,\frac{y_n(\tau)}{y_n(\tau_1)}\right]\sin(n\theta+\mu_n),$$

whence

$$\varphi = -\frac{2\tau}{(1-\tau)^{\beta}}\sum_n B_n\left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n}\left[\frac{1}{n}\,\frac{y_n'(\tau)}{y_n(\tau_1)} + \frac{1}{2\tau}\,\frac{y_n(\tau)}{y_n(\tau_1)}\right]\cos(n\theta+\mu_n) + F(\tau)$$

where, by (2.9) $F(\tau)$ is such that

$$\varphi_\tau = \frac{(2\beta+1)\tau-1}{2\tau(1-\tau)^{\beta+1}}\,\psi_\theta$$

and ψ_θ is derived from (2.14). Thus we find that

$$F(\tau) = B\int_{\tau_1}^{\tau}\frac{(2\beta+1)\tau-1}{2\tau(1-\tau)^{\beta+1}}\,d\tau$$

and

$$\varphi(\theta,\tau) = B\int_{\tau_1}^{\tau}\frac{(2\beta+1)\tau-1}{2\tau(1-\tau)^{\beta+1}}\,d\tau - (1-\tau)^{-\beta}\sum_n B_n\left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n} x_n(\tau)\,\frac{y_n(\tau)}{y_n(\tau_1)}$$
$$\times\cos(n\theta+\mu_n), \qquad (2.15)$$

where

$$x_n(\tau) = 1 + \frac{2\tau}{n}\,\frac{y_n'(\tau)}{y_n(\tau)}.$$

Let us now determine what (2.14) and (2.15) reduce to when the fluid is incompressible. q_m is now infinite, τ and τ_1 are both zero although τ/τ_1 is replaced by $(q/q_1)^2$, $y_n(0)=1$, $y_n'(0) = -\frac{1}{2}n\beta$ and

$\int_{\tau_1}^{\tau}\frac{(2\beta+1)\tau-1}{2\tau(1-\tau)^{\beta+1}}\,d\tau$ reduces to $-\log(q/q_1)$.

Hence, if $A = 0$,

$$\psi(\theta,\tau) = B\theta + \sum_n B_n\left(\frac{q}{q_1}\right)^n\sin(n\theta+\mu_n), \qquad (2.16)$$

$$\varphi(\theta,\tau) = -B\log q/q_1 - \sum_n B_n\left(\frac{q}{q_1}\right)^n\cos(n\theta+\mu_n).$$

i.e., if the phases of the circular functions are taken as zero,

$$w \equiv \varphi + i\psi = -B\log\left(\frac{q}{q_1}e^{-i\theta}\right) - \sum_n B_n\left(\frac{q}{q_1}e^{-i\theta}\right)^n. \qquad (2.17)$$

Thus, following Ringleb, we see that equations (2.16) are completely equivalent to the expansion of the complex potential function w in powers of $\frac{q}{q_1} e^{-i\theta}$ i.e. in powers of $\frac{1}{q_1}\frac{dw}{dz}$ since

$$\frac{dw}{dz} = \frac{\partial}{\partial x}(\varphi + i\psi) = u - iv = q e^{-i\theta}. \qquad (2.18)$$

Hence, if the complex potential function is known for a field of incompressible flow and is expanded in the form (2.17), the coefficients B_1 B_n so determined can be substituted in (2.14) (2.15) to determine the velocity potential and stream function for a correspo-nding field of compressible flow.

The question of the convergence of the series (2.14) and (2.15) will be discussed in the next section.

The solution of equations (2.14) and (2.15) gives φ and ψ in terms of θ and τ. On the streamline $\psi =$ const., we therefore have

$$\psi(\theta, \tau) = const. \qquad (2.19)$$

In order to determine the flow in the xy plane we then have recourse to the relationship

$$dz \equiv dx + i\,dy = q^{-1} e^{i\theta}\left(d\varphi + i \frac{\rho_0}{\rho} d\psi\right) \qquad (2.20)$$

which follows easily from equations (1.14) and (1.15) defining the velocity components in terms of the derivatives of φ and ψ. Hence, eliminating θ and τ between (2.19) and the two equations obtained from (2.20) by equating real and imaginary parts, we obtain the x-y equation of the streamline $\psi(\theta, \tau) =$ const.

III The Convergence of the Series for φ and ψ in the Hodograph Method, and the Determination of Approximate Functions for φ and ψ.

It has been proved by Jacob, on the basis of a Riccati equation, which is satisfied by $x_n(\tau)$, that the series (2.14) and (2.15) giving ψ and φ for the compressible flow converge uniformly and absolutely in the range $0 \leqq \tau \leqq \tau_1$ provided the same is true of the series (2.16) for the corresponding incompressible flow and that the series are in ascending powers of τ/τ_1. We shall now show how Jacob's argument can be extended to prove that this result is valid in the wider range $0 \leqq \tau \leqq \frac{1}{2\beta+1}$. Physically, this is much more satisfactory, for the critical velocity given by $\tau = (2\beta+1)^{-1}$ is of much greater significance than the main stream velocity given by $\tau = \tau_1$.

We begin by proving three lemmas due to Jacob and take $n > 0$ throughout.

Lemma I In the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$, $0 < y_n(\tau) \leqq 1$ and $0 < x_n(\tau) \leqq 1$.

Since $y_n(\tau)$ satisfies the equation (2.13)

$$\tau(1-\tau)y_n'' + \left[n+1-\tau(n-\beta+1)\right]y_n' + \tfrac{1}{2}n(n+1)\beta y_n = 0,$$

it follows that

$$\frac{d}{d\tau}\left[\tau(1-\tau)^{-\beta} z_n'(\tau)\right] - \frac{n^2}{4}\,\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}\, z_n(\tau) = 0\,, \tag{3.1}$$

where $$z_n(\tau) \equiv \tau^{\frac{1}{2}n} y_n(\tau).$$

Let $\tau = a$ be the first zero of $y_n(\tau)$ in the interval considered. Then $z_n(\tau)$ vanishes for $\tau = 0$ and $\tau = a$ and so, by Rolle's Theorem, there exists a value b where $0 < b < a$ for which $z_n'(\tau) = 0$. The function $\tau(1-\tau)^{-\beta} z_n'(\tau)$ is therefore zero for $\tau = 0$ and $\tau = b$ and so, by Rolle's Theorem, its derivative vanishes for some value $\tau = c$, where $0 < c < b$. Hence, by (3.1), $z_n(\tau)$ vanishes when $\tau = c$ $(< a)$, and this contradicts the initial assumption. Hence $z_n(\tau)$ and $y_n(\tau)$ have no zeros in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$. Since $y_n(0) = 1$ it follows that $y_n(\tau) > 0$ and $z_n(\tau) \geqq 0$ in the range considered.

Further, $z_n'(\tau)$ cannot have a zero in the interval, otherwise by (3.1), $z_n(\tau)$ would have a zero in the interval. Now

$$z_n'(\tau) = \frac{n}{2\tau} z_n(\tau)\, x_n(\tau)$$

and $z_n(\tau)$ and $x_n(\tau)$ are both positive in the neighbourhood of $\tau = 0$. Hence $z_n'(\tau) > 0$ in the neighbourhood of $\tau = 0$ and therefore $z_n'(\tau) > 0$

in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$. Since $z_n(\tau) > 0$ throughout the interval, it follows that the same is true of $x_n(\tau)$,

$$\text{i.e.} \qquad x_n(\tau) > 0, \qquad 0 \leqq \tau \leqq (2\beta+1)^{-1}.$$

Also, (2.13) may be written

$$\frac{d}{d\tau}\left[\tau^{n+1}(1-\tau)^{-\beta} y_n'(\tau)\right] + \tfrac{1}{2} n\beta(n+1)\tau^{n}(1-\tau)^{-\beta-1} y_n(\tau) = 0.$$

By similar reasoning to the above, it follows that $y_n'(\tau)$ cannot be zero in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$. Now $y_n'(0) = -\frac{1}{2}n\beta < 0$, whence $y_n'(\tau) < 0$, and, since $y_n(0) = 1$, $y_n(\tau) \leqq 1$. Thus $0 < y_n(\tau) \leqq 1$, $0 \leqq \tau \leqq (2\beta+1)^{-1}$.

It follows too that $x_n(\tau) \leqq 1$ and so, by the above

$$0 < x_n(\tau) \leqq 1, \qquad 0 \leqq \tau \leqq (2\beta+1)^{-1}.$$

The Riccati Equation for $x_n(\tau)$.

From the definition of $x_n(\tau)$ we immediately have that

$$2\tau y_n'(\tau) = n y_n(\tau)\left[x_n(\tau) - 1\right] \tag{3.2}$$

whence $2\left[y_n''(\tau) + y_n'(\tau)\right] = n y_n'(\tau)\left[x_n(\tau) - 1\right] + n x_n'(\tau) y_n(\tau).$

Substituting for $y_n''(\tau)$ in terms of $y_n'(\tau)$ and $y_n(\tau)$ (Cf. (2.13)), and then for $y_n(\tau)$ in terms of $y_n'(\tau)$ from (3.2) and dividing throughout by $y_n'(\tau)$, we readily find that $x_n(\tau)$ satisfies the Riccati equation

$$F_n(x) \equiv \tau(1-\tau)x_n'(\tau) + \tfrac{1}{2}n(1-\tau)x_n^2(\tau) + \beta\tau x_n(\tau) - \tfrac{n}{2}\left[1-(2\beta+1)\tau\right] = 0. \tag{3.3}$$

Lemma II Let $k(\tau)$ be a continuous function of τ with continuous first derivative in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$, where $k(\tau) \geqq 0$ and $k(0) = 1$. If $F_n[k(\tau)] \geqq 0$ in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$, then $k(\tau) \geqq x_n(\tau)$, in the same interval; similarly if $F_n[k(\tau)] \leqq 0$, then $k(\tau) \leqq x_n(\tau)$, $0 \leqq \tau \leqq (2\beta+1)^{-1}$.

Let $F_n[k(\tau)] \geqq 0$, then

$$F_n[k(\tau)] - F_n[x_n(\tau)] \equiv \tau(1-\tau)[k'(\tau) - x_n'(\tau)]$$
$$+ \left\{\tfrac{1}{2}n(1-\tau)[k(\tau)+x_n(\tau)] + \beta\tau\right\}[k(\tau) - x_n(\tau)] \geqq 0. \quad (3.4)$$

Now suppose that $k(\tau) < x_n(\tau)$ at some point in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$. Then $k(\tau) - x_n(\tau)$ must have a negative lower bound at some point $\tau = a$ in the interval. If $\tau = a$ is an interior point of the interval, it must correspond to a turning value of $k(\tau) - x_n(\tau)$ and $k'(a) - x_n'(a) = 0$. Hence, since the curly bracket in (3.4) is positive by Lemma I, we must have

$$k(a) - x_n(a) \geqq 0$$

which contradicts the fact that the lower bound is negative.

If $\tau = a$ is an end point of the interval, then $a = (2\beta+1)^{-1}$ since $k(0) - x_n(0) = 0$. Also, $k'(a) - x_n'(a) \leqq 0$, whence

$$k(a) - x_n(a) \geqq 0,$$

which is again a contradiction.

Hence $k(\tau) \geqq x_n(\tau)$ throughout the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$ and, we can prove in a similar manner that, if $F_n[k(\tau)] \leqq 0$, then $k(\tau) \leqq x_n(\tau)$ throughout the same interval.

Lemma II can be used to show that

$$x_n^2(\tau) \geqq 1 - \frac{2\beta\tau}{1-\tau}. \tag{3.5}$$

Let $k(\tau) \equiv \left(1 - \frac{2\beta\tau}{1-\tau}\right)^{1/2}$, then $k(\tau)$ satisfies the conditions of the lemma and

$$F_n[k(\tau)] = -\frac{\beta(2\beta+1)\tau^2}{(1-\tau)\left(1-\frac{2\beta\tau}{1-\tau}\right)^{1/2}} \leqq 0.$$

Hence $k(\tau) \leqq x_n(\tau)$ and the result follows.

<u>Lemma III</u> $x_n(\tau)$ is a decreasing function of n in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$.

Let $p > 0$, then

$$F_n[x_{n+p}(\tau)] = F_n[x_{n+p}(\tau)] - F_{n+p}[x_{n+p}(\tau)]$$
$$= -\tfrac{1}{2}p\left[(1-\tau)x_{n+p}^2(\tau) - \{1-(2\beta+1)\tau\}\right]$$

by (3.3). Hence by (3.5),

$$F_n[x_{n+p}(\tau)] \leqq 0,$$

and by Lemma II,

$$x_{n+p}(\tau) \leqq x_n(\tau).$$

The whole of the foregoing was proved by Jacob who, after proving two other lemmas, derived the result stated at the beginning of this section. However, if we proceed from this point in a rather different way, we can establish the more general result to which we referred.

We continue to regard n as positive and realise that the Riccati equation (3.3) can be solved exactly in two special cases, viz. $n = 0$, $n \to \infty$.

When $n = 0$, we have

$$\tau(1-\tau)\, x_0'(\tau) + \beta\tau\, x_0(\tau) = 0$$

whence $\quad x_0(\tau) = K(1-\tau)^\beta$, $\quad K$ a constant.

Since $x_0(0) = 1$ (from the definition), $K = 1$ and

$$x_0(\tau) = (1-\tau)^\beta. \tag{3.6}$$

When $n \to \infty$, $x_n(\tau)$ and $x_n'(\tau)$ remain finite (cf. Lemma I) and, $x_n(\tau) \to x_\infty(\tau)$, where by (3.3)

$$(1-\tau)\, x_\infty^2(\tau) = \{1-(2\beta+1)\tau\},$$

i.e. $$x_\infty(\tau) = \left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2}, \tag{3.7}$$

the positive sign being taken to ensure that $x_\infty(0) = 1$.

Thus, by Lemma III

$$\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2} \leqq x_n(\tau) \leqq (1-\tau)^\beta, \tag{3.8}$$

or $$\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2} \leqq 1 + \frac{2\tau}{n}\frac{y_n'(\tau)}{y_n(\tau)} \leqq (1-\tau)^\beta$$

It follows from this inequality that if $\tau > \tau_1$

$$\frac{n}{2}\int_{\tau_1}^{\tau}\left\{\frac{1}{\tau}\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{\frac{1}{2}} - \frac{1}{\tau}\right\}d\tau \leqq \log\frac{y_n(\tau)}{y_n(\tau_1)} \leqq \frac{n}{2}\int_{\tau_1}^{\tau}\frac{(1-\tau)^\beta-1}{\tau}d\tau. \tag{3.9}$$

The integrals in this inequality are both negative when $\tau \geqq \tau_1$, $0 \leqq \tau \leqq (2\beta+1)^{-1}$, hence

$$\log \frac{y_n(\tau)}{y_n(\tau_1)} \leqq 0$$

and $$\frac{y_n(\tau)}{y_n(\tau_1)} \leqq 1. \qquad (3.10)$$

Thus

$$\left| B_n \left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n} \frac{y_n(\tau)}{y_n(\tau_1)} \sin(n\theta + \mu_n) \right| \leqq \left| B_n \left(\frac{\tau}{\tau_1}\right)^{\frac{1}{2}n} \sin(n\theta + \mu_n) \right|,$$

whence it follows from the general principle of convergence that the series (2.14) is absolutely and uniformly convergent in the interval $0 \leqq \tau_1 \leqq \tau \leqq (2\beta+1)^{-1}$, provided the same is true of the corresponding series (2.16).

If $\tau < \tau_1$, the inequality (3.9) is reversed and becomes

$$\frac{n}{2}\int_{\tau_1}^{\tau} \frac{(1-\tau)^{\beta}-1}{\tau}\,d\tau \leqq \log \frac{y_n(\tau)}{y_n(\tau_1)} \leqq \frac{n}{2}\int_{\tau_1}^{\tau} \left\{ \frac{1}{\tau}\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{\frac{1}{2}} - \frac{1}{\tau}\right\} d\tau. \qquad (3.11)$$

Hence

$$\frac{n}{2}\int_{\tau_1}^{\tau} \frac{(1-\tau)^{\beta}\,d\tau}{\tau} \leqq \log \frac{\tau^{\frac{1}{2}n}\, y_n(\tau)}{\tau_1^{\frac{1}{2}n}\, y_n(\tau_1)} \leqq \frac{n}{2}\int_{\tau_1}^{\tau} \frac{1}{\tau}\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{\frac{1}{2}} d\tau.$$

Both integrals are again negative or zero since $0 \leqq \tau \leqq \tau_1 \leqq (2\beta+1)^{-1}$, so that

$$\frac{\tau^{1/2 n}\, y_n(\tau)}{\tau_1^{1/2 n}\, y_n(\tau_1)} \leqq 1. \qquad (3.12)$$

Now, if the series (2.16) is absolutely convergent in the interval $0 \leqq \tau \leqq \tau_1$ (i.e. in $0 \leqq q/q_1 \leqq 1$) then the series $\sum \left| B_n \sin(n\theta + \mu_n) \right|$ is convergent. Hence by (3.12) it follows from the M-Test that the series (2.14) is uniformly and absolutely convergent in the interval $0 \leqq \tau \leqq \tau_1 \leqq (2\beta+1)^{-1}$.

Thus, our result holds whether $\tau \leqq \tau_1$, or $\tau \geqq \tau_1$ i.e. the series (2.14) is absolutely and uniformly convergent in the interval $0 \leqq \tau \leqq (2\beta+1)^{-1}$ provided the same is true of the corresponding series (2.16).

By (3.8), $|x_n(\tau)| < 1$ so that the same result holds for (2.15), the series expansion of $\varphi(\theta,\tau)$.

Approximations for $y_n(\tau)$ and $x_n(\tau)$.

The inequality (3.8) and a similar one which will be given below enable us to obtain fairly close approximations to $y_n(\tau)$ and $x_n(\tau)$. This method of approximation will be at least as good, if not better, than that due to Karman and Tsien.[1]

Giving β the value $5/2$, it follows from (3.8) that

$$1 - \frac{5}{2}\tau - \frac{45}{8}\tau^2 \dots \leqq x_n(\tau) \leqq 1 - \frac{5}{2}\tau + \frac{15}{8}\tau^2 \dots ,$$

i.e.

$$\frac{n}{2}\left(-\frac{5}{2} - \frac{45}{8}\tau \cdots\right) \leqq \frac{y_n'(\tau)}{y_n(\tau)} \leqq \frac{n}{2}\left(-\frac{5}{2} + \frac{15}{8}\tau \dots\right).$$

Hence, if $\tau > \tau_1$

$$\frac{n}{2}\left(-\frac{5}{2}\tau - \frac{45}{16}\tau^2 \cdots\right) \leqq \log y_n(\tau) \leqq \frac{n}{2}\left(-\frac{5}{2}\tau + \frac{15}{16}\tau^2 \dots\right),$$

the constant of integration being taken as zero to ensure that $y_n(0) = 1$; if $\tau < \tau_1$, the inequality is reversed.

Since $0 \leqq \tau \leqq 1/6$, we have approximately (in

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1. Tsien: Journ. Aero. Sci. 6 (1939), 399-407.

both cases)

$$\log y_n(\tau) = \frac{n}{2}\left(-\frac{5}{2}\tau - \frac{15}{16}\tau^2\right),$$

whence

$$y_n(\tau) \doteqdot \left[\exp\left\{-\frac{5}{4}\tau - \frac{15}{32}\tau^2\right\}\right]^n. \tag{3.13}$$

In estimating the error incurred by the approximation, let us remember that after we have found $y_n(\tau)$ we determine the stream function for the compressible flow by replacing the term $(q/q_1)^n$ in the first of equations (2.16) by $\left(\frac{q}{q_1}\right)^n \frac{y_n(\tau)}{y_n(\tau_1)}$. Hence, if we take the expression (3.13) for $y_n(\tau)$, we replace $\frac{q}{q_1}$ by $\frac{q}{q_1} \frac{\exp\{-\frac{5}{4}\tau - \frac{15}{32}\tau^2\}}{\exp\{-\frac{5}{4}\tau_1 - \frac{15}{32}\tau_1^2\}}$. Thus, although our approximation may lead to an error of 12% (when $\tau = 1/6$) in the index of each exponential, the resultant error in q/q_1 for the compressible flow is at most only $\frac{5}{4}\cdot\frac{1}{6}\cdot 24$ i.e. 5%. The approximation can therefore be regarded as satisfactory.

To enable us to calculate $\varphi(\theta, \tau)$ we now seek an approximation to $x_n(\tau)$. The inequality (3.8) is not sufficiently strong, so we try to find a function related to the functions in the series (2.15) for $\varphi(\theta, \tau)$ in the same way as $x_n(\tau)$ is related to the functions appearing in the series (2.14) for $\psi(\theta, \tau)$ and then repeat the above procedure.

The function required is $z_n(\tau)$, where

$$\mathfrak{z}_n(\tau) = 1 + \frac{2\tau}{n} \frac{\{(1-\tau)^{-\beta} x_n(\tau) y_n(\tau)\}'}{\{(1-\tau)^{-\beta} x_n(\tau) y_n(\tau)\}}.$$

Simplifying, we easily find that this reduces to

$$\mathfrak{z}_n(\tau) = \frac{1-(2\beta+1)\tau}{(1-\tau)\, x_n(\tau)}.$$

Inverting and differentiating, we find that

$$x_n'(\tau) = -\frac{1-(2\beta+1)\tau}{1-\tau} \frac{\mathfrak{z}_n'(\tau)}{\mathfrak{z}_n(\tau)} - \frac{1}{\mathfrak{z}_n(\tau)} \frac{2\beta}{(1-\tau)^2},$$

whence, substituting for $x_n(\tau)$ and $x_n'(\tau)$ in (3.3), we see that $\mathfrak{z}_n(\tau)$ satisfies the Riccati equation

$$\mathfrak{z}_n'(\tau) + \frac{\beta\{1+(2\beta+1)\tau\}}{(1-\tau)\{1-(2\beta+1)\tau\}} \mathfrak{z}_n(\tau) + \frac{n}{2\tau} \mathfrak{z}_n^2(\tau) - \frac{1}{2} n \frac{1-(2\beta+1)\tau}{\tau(1-\tau)} = 0. \tag{3.14}$$

As before, this equation can be solved exactly in the cases $n=0$ and $n\to\infty$; we have

$$\mathfrak{z}_\infty(\tau) = \left\{\frac{1-(2\beta+1)\tau}{1-\tau}\right\}^{1/2}$$

and

$$\frac{\mathfrak{z}_0'(\tau)}{\mathfrak{z}_0(\tau)} = \frac{\beta+1}{1-\tau} - \frac{2\beta+1}{1-(2\beta+1)\tau},$$

i.e.

$$\mathfrak{z}_0(\tau) = \frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}},$$

the constant of integration being chosen to make $\mathfrak{z}_0(0) = 1$.

The equation (3.14) has the same properties as (3.3), the argument of Lemma II still being valid, <u>mutatis mutandis</u>.

Hence

$$\left\{\frac{1-(2\beta+1)\tau}{1-\tau}\right\}^{1/2} \leqq \mathfrak{z}_n(\tau) \leqq \frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}},$$

i.e.

$$1-\frac{5}{2}\tau-\frac{45}{8}\tau^2\ldots \leqq 1+\frac{2\tau}{n}\frac{\{(1-\tau)^{-\beta}x_n(\tau)\,y_n(\tau)\}'}{\{(1-\tau)^{-\beta}x_n(\tau)\,y_n(\tau)\}} \leqq 1-\frac{5}{2}\tau-\frac{105}{8}\tau^2\ldots.$$

As our approximation we take

$$\frac{2\tau}{n}\frac{\{\ \}'}{\{\ \}} = -\frac{5}{2}\tau-\frac{75}{8}\tau^2\ ;$$

the error intoduced here is about the same as in the previous case. Thus

$$(1-\tau)^{-\beta}x_n(\tau)\,y_n(\tau) \doteq \left[\exp\left(-\frac{5}{4}\tau-\frac{75}{32}\tau^2\right)\right]^n. \tag{3.15}$$

Hence, by (3.13) (3.15) we have the rule:

to obtain the series (2.15) from the correspondi ng series (2.16) replace q/q_1 in (2.16) by

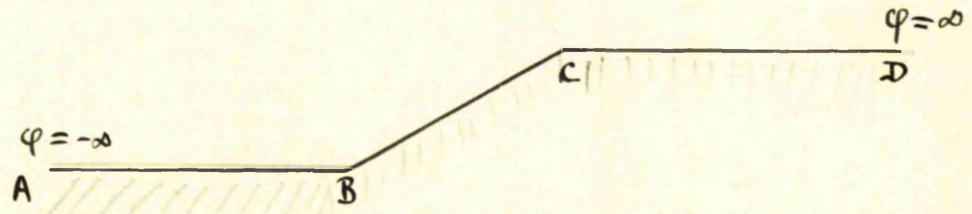
$$\frac{q}{q_1}\,\frac{\exp\left(-\frac{5}{4}\tau-\frac{75}{32}\tau^2\right)}{\exp\left(-\frac{5}{4}\tau_1-\frac{15}{32}\tau_1^2\right)}.$$

In the next section, we shall apply these formulae to one or two problems.

IV. Some applications of the Hodograph Method

In the solution of any problem by the hodograph method, the first step is to expand the complex potential function $w \equiv \varphi + i\psi$ in powers of $\frac{q}{q_1}e^{-i\theta}$. Now this latter function is extremely useful (as an auxiliary variable) in the solution of problems relating to free jets or flow along rectilinear boundaries, because q is constant on the surface of a free jet, while θ is constant along any boundary[1]. It is natural therefore to try to apply the hodograph method to such problems as these. Problems on free jets have been extensively treated by Tchapliguine, Jacob, Villat and others but, as far as I am aware, there is no mention in the literature of any problem concerning flow along rectilinear boundaries. We therefore attempt to deal with one or two such problems here.

Flow of a deep stream over a rectilinear bed



1. See, for example, Ramsey: Hydromechanics 5th ed. Part II, Chapter 6.

We consider a fluid flowing from A to D which are infinitely distant from and on either side of the sloping part BC of the boundary ABCD; the fluid is assumed to extend to infinity on the upper side of the boundary.

We begin by assuming that the fluid is incompressible.

Let BC be of length l and let $\hat{ABC} = \pi - \alpha$. Let the velocity at infinity be U and let us take the velocity potential φ as $-\infty$ at A as $+\infty$ at D, since the direction of flow is the direction of increasing velocity potential.

This particular field of flow in the z-plane can be obtained by means of the Schwarz-Christoffel transformation from the uniform rectilinear flow parallel to the real axis in another plane - the ζ plane. The Theorem of Schwarz and Christoffel, defining the transformation may be stated as follows[1].

Let $a, b, c, \ldots\ldots$ be n points on the real axis in the ζ plane such that $a < b < c < \ldots\ldots$ and let $\alpha, \beta, \gamma, \ldots\ldots$ be the interior angles of a simple closed polygon of n vertices so that $\alpha + \beta + \gamma + \ldots\ldots = (n-2)\pi$. Then the transformation from the ζ-plane to the z-plane, defined by

$$\frac{dz}{d\zeta} = K(\zeta - a)^{\frac{\alpha}{\pi}-1}(\zeta - b)^{\frac{\beta}{\pi}-1}(\zeta - c)^{\frac{\gamma}{\pi}-1}\ldots \qquad (4.1)$$

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1. See Milne-THOMSON: Theoretical Hydrodynamics, p. 258

transforms the real axis in the ζ- plane into the boundary of a closed polygon in the z-plane in such a way that the vertices of the polygon correspond to the points $a, b, c, \ldots\ldots$ and the interior angles of the polygon are $\alpha, \beta, \gamma, \ldots\ldots$. Moreover, when the polygon is simple, the interior is mapped by the transformation on the upper half of the ζ-plane.

The constant K may be complex and all polygons corresponding to given values of $a, b, c, \ldots$ $\alpha, \beta, \gamma, \ldots\ldots$ are similar. If three of the numbers $a, b, c, \ldots\ldots$ are chosen arbitrarily to correspond to three of the vertices of a given polygon, the remainder must then be arranged to make the polygon the proper shape. It can be shown, too, that when a vertex of the polygon corresponds to a point at infinity on the ζ-axis, the corresponding factor in (4.1) should be omitted; the angle of the polygon at the vertex concerned does not then appear.

Reverting now to the problem under consideration, let A, B, C, D correspond to $\zeta = -\infty, 0, a, \infty$ respectively. Since we can choose only three of these values arbitrarily, we assign to the point C

the corresponding point $\zeta = a$; a depends on the value of l. The Schwarz-Christoffel Theorem then gives
$$\frac{dz}{d\zeta} = K\zeta^{\frac{\pi-\alpha}{\pi}-1}(\zeta-a)^{\frac{\pi+\alpha}{\pi}-1},$$
where K is a constant,

i.e.
$$\frac{dz}{d\zeta} = K\left(\frac{\zeta-a}{\zeta}\right)^{\alpha/\pi}. \tag{4.2}$$

We have here regarded the fluid as enclosed in a polygon in the z- plane with two vertices at infinity, and we shall determine the flow in the z-plane which corresponds to uniform rectilinear flow in the ζ- plane. Putting it rather crudely, we are considering a uniform rectilinear flow along the real axis of the ζ- plane and then deforming the boundary to the shape ABCD in the z-plane. The velocities at points infinitely distant from BC are unaltered by the transformation.

In the ζ-plane, the complex potential function w is given by
$$w \equiv \varphi + i\psi = U\zeta \tag{4.3}$$
whence, by (4.2)
$$U\frac{dz}{dw} = K\left(1-\frac{aU}{w}\right)^{\alpha/\pi}.$$
At an infinite distance from BC, $w = \infty$ and $dz/dw = 1/U$, so that K = 1. Thus
$$U\frac{dz}{dw} = \left(1-\frac{aU}{w}\right)^{\alpha/\pi}. \tag{4.4}$$

Now, by (4.2) and (4.3),

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = U\left(1-\frac{a}{\zeta}\right)^{-\alpha/\pi},$$

from which it follows that dw/dz is infinite when $\zeta = a$, i.e. the fluid has infinite velocity at the point corresponding to $\zeta = a$, i.e. at C. When we pass on to the case of a compressible fluid, the flow in the neighbourhood of C will be very interesting, since there is an upper limit to the velocity which can be attained by a compressible fluid.

If we take the boundary ABCD as the streamline $\psi = 0$, then $w \equiv \varphi$ on ABCD and (4.3) shows that

$$w \equiv \varphi = Ua \qquad \text{at C}$$

$$w \equiv \varphi = 0 \qquad \text{at B.}$$

We are now in a position to determine the relationship between l and a .

On BC,

$$z = \frac{1}{U}\int\left(1-\frac{aU}{\varphi}\right)^{\alpha/\pi} d\varphi,$$

by (4.4), whence, taking B as the origin in the z- plane and integrating from B to C, we have

$$le^{i\alpha} = \frac{1}{U}\int_0^{Ua}(-1)^{\alpha/\pi}\left(-1+\frac{aU}{\varphi}\right)^{\alpha/\pi} d\varphi$$

or
$$l = a\int_0^1 x^{-\alpha/\pi}(1-x)^{\alpha/\pi}\,dx$$

where $\varphi = Uax$. Hence, using the well-known properties of the B and Γ functions, we find that

$$l = a\,B\left(1-\frac{\alpha}{\pi},\ 1+\frac{\alpha}{\pi}\right)$$

$$= a\,\Gamma\left(1-\frac{\alpha}{\pi}\right)\Gamma\left(1+\frac{\alpha}{\pi}\right)$$

$$= a\,\frac{\alpha}{\pi}\,\Gamma\left(1-\frac{\alpha}{\pi}\right)\Gamma\left(\frac{\alpha}{\pi}\right),$$

i.e. $l = \dfrac{a\alpha}{\sin\alpha}$, or $\qquad \dfrac{l}{a} = \dfrac{\alpha}{\sin\alpha}.$ (4.5)

The flow is completely determined by (4.4) and (4.5).

Now, let q denote the resultant fluid velocity at any point in the field and let θ be the angle which it makes with the real axis. Then

$$\frac{dw}{dz} = q e^{-i\theta},$$

and

$$w = \frac{aU}{1-\left(\frac{U}{q e^{-i\theta}}\right)^{\pi/\alpha}}. \tag{4.6}$$

We are specially interested in the part of the field of flow for which $q > U$ (near C). For this region

$$\varphi + i\psi = aU\sum_{0}^{\infty}\left(\frac{q e^{-i\theta}}{U}\right)^{-n\pi/\alpha},$$

so that (4.7)

$$\varphi = aU\left\{1+\sum_{1}^{\infty}\left(\frac{q}{U}\right)^{-n\pi/\alpha}\cos\frac{n\pi\theta}{\alpha}\right\},$$

$$\psi = aU\sum_{1}^{\infty}\left(\frac{q}{U}\right)^{-n\pi/\alpha}\sin\frac{n\pi\theta}{\alpha}. \tag{4.8}$$

We now pass on to the case of a compressible fluid and use the hodograph method to determine φ and ψ. We shall consider the special case in which the Mach number M of the undisturbed stream is 0·8 and $\alpha = 45^0$, and shall determine the sonic line of the flow, i.e. the locus of points at which M is unity.

We note by (1.11) that $\tau_1 = 0{\cdot}1134$ and that $\tau = 1/6$ on the sonic line.

Now, by (3.13) and (3.15), the stream function and velocity potential for the compressible flow are obtained from (4.7) and (4.8) which apply to the incompressible flow by replacing $\frac{q}{u}$ by

$\left(\frac{\tau}{\tau_1}\right)^{1/2} \dfrac{\exp\left(-\frac{5}{4}\tau - \frac{15}{32}\tau^2\right)}{\exp\left(-\frac{5}{4}\tau_1 - \frac{15}{32}\tau_1^2\right)}$ in the case of ψ and by

$\left(\frac{\tau}{\tau_1}\right)^{1/2} \dfrac{\exp\left(-\frac{5}{4}\tau - \frac{75}{32}\tau^2\right)}{\exp\left(-\frac{5}{4}\tau_1 - \frac{15}{32}\tau_1^2\right)}$ in the case of φ, i.e.

we replace q/u in the expansion of ψ by 1·126 and in the expansion of φ by 1·079.

Hence, on the sonic line

$$\psi = aU\sum_1^\infty (1{\cdot}126)^{-4n}\sin 4n\theta \,, \tag{4.9}$$

$$\varphi = C + aU\sum_1^\infty (1{\cdot}079)^{-4n}\cos 4n\theta \,, \tag{4.10}$$

where C is a constant, whose value we shall not require to determine.

Now by (2.20) and 1.19)

$$dz = q^{-1}e^{i\theta}\left\{d\varphi + i(1-\tau)^{-\beta}d\psi\right\}$$

so that, on the sonic line

$$\frac{dz}{d\theta} = \frac{e^{i\theta}}{c}\left\{\frac{d\varphi}{d\theta} + i\left(\frac{5}{6}\right)^{-5/2}\frac{d\psi}{d\theta}\right\}.$$

Thus, by (4.9) and (4.10)

$$\frac{dz}{d\theta} = \frac{aU}{c}e^{i\theta}\sum_{1}^{\infty}\left\{-4n(1\cdot079)^{-4n}\sin 4n\theta + 4in\left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\cos 4n\theta\right\}.$$

Also, by (1.10),

$$\frac{U}{c} = 0\cdot8\left(\frac{1-\tau_1}{1-\tau}\right)^{1/2} = 0\cdot8254,$$

and, by (4.5)

$$a = \frac{2\sqrt{2}\,l}{\pi},$$

Whence

$$\frac{dz}{d\theta} = \frac{0\cdot8254.\,2\sqrt{2}\,l}{\pi}\sum_{1}^{\infty}4n\Big[\left\{-(1\cdot079)^{-4n}\sin 4n\theta\cos\theta - \left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\sin\theta\cos 4n\theta\right\}$$
$$+ i\left\{-(1\cdot079)^{-4n}\sin 4n\theta\sin\theta - \left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\cos\theta\cos 4n\theta\right\}\Big]$$

Integrating, we obtain

$$z = \frac{aU}{c}\sum_{1}^{\infty}2n\Big[(1\cdot079)^{-4n}\left\{\frac{\cos(4n+1)\theta}{4n+1} + \frac{\cos(4n-1)\theta}{4n-1}\right\} + \left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\left\{\frac{\cos(4n+1)\theta}{4n+1} - \frac{\cos(4n-1)\theta}{4n-1}\right\}$$

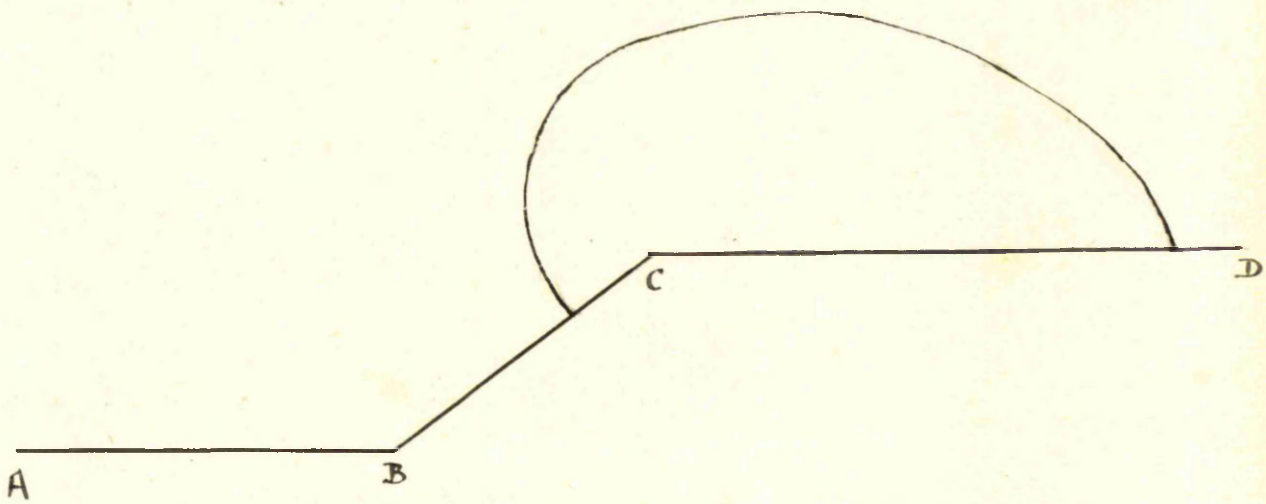
$$-i(1\cdot079)$$

and, on equating real and imaginary parts, have the freedom equations of the sonic line:

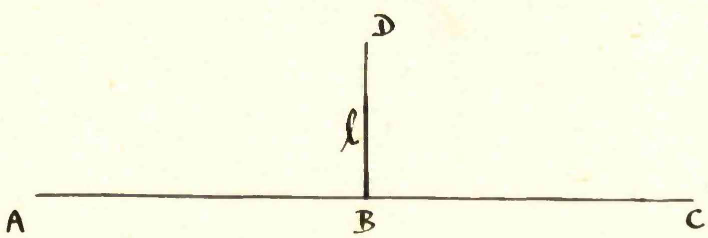
$$x = 1\cdot4864\,l\sum_{1}^{\infty}n\Big[(1\cdot079)^{-4n}\left\{\frac{\cos(4n+1)\theta}{4n+1} + \frac{\cos(4n-1)\theta}{4n-1}\right\} \qquad (4.11)$$
$$+ \left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\left\{\frac{\cos(4n+1)\theta}{4n+1} - \frac{\cos(4n-1)\theta}{4n-1}\right\}\Big],$$

$$y = 1\cdot4864\,l\sum_{1}^{\infty}n\Big[(1\cdot079)^{-4n}\left\{\frac{\sin(4n+1)\theta}{4n+1} - \frac{\sin(4n-1)\theta}{4n-1}\right\} \qquad (4.12)$$
$$+ \left(\frac{5}{6}\right)^{-5/2}(1\cdot126)^{-4n}\left\{\frac{\sin(4n+1)\theta}{4n+1} + \frac{\sin(4n-1)\theta}{4n-1}\right\}\Big].$$

The sonic line can now be drawn by calculating x and y various values of θ in the range $0 \leq \theta \leq 45^\circ$. By (4.12) $y = 0$ when $\theta = 0$, so the origin of co-ordinates lies somewhere on CD. A rough sketch is shown in the diagram.

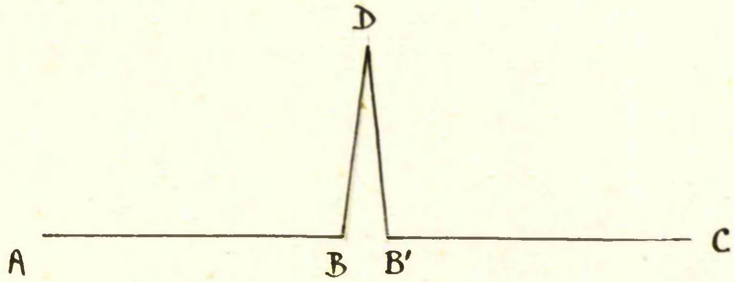


Flow of a wide stream past a thin obstacle projecting perpendicular to a straight bank.



Let the obstacle be of length ℓ and let the fluid, assumed for the moment to be incompressible, flow from

A to C which are infinitely distant from and on either side of the obstacle BD. Let the velocity at infinity be U. We consider the fluid as occupying the interior of the polygon.....ABDB$'$C.... on which B, B$'$ are zero distance apart, and, taking B as the origin



in the z-plane, map the polygon on to the ζ-plane with the following correspondence

$$A \to \zeta = -\infty, \quad B \to \zeta = -l, \quad D \to \zeta = 0,$$
$$B' \to \zeta = l, \quad C \to \zeta = \infty.$$

By Schwarz-Christoffel theorem, the transformation is defined by

$$\frac{dz}{d\zeta} = K(\zeta + l)^{-1/2}\,\zeta\,(\zeta - l)^{-1/2} = \frac{K\zeta}{\sqrt{(\zeta^2 - l^2)}},$$

whence $\quad z = K\sqrt{(\zeta^2 - l^2)} + \text{constant}.$

At B, $z = 0$ and $\zeta = -l$, so that the constant of integration vanishes and

$$z = K\sqrt{(\zeta^2 - l^2)}.$$

At D, $z = il$ and $\zeta = 0$, hence $K = 1$ and

$$z = \sqrt{(\zeta^2 - l^2)}. \qquad (4.13)$$

Now, as before, the flow in the z-plane is obtained from a uniform rectilinear flow in the ζ-plane, so that

$$w = U\zeta \tag{4.14}$$

By (4.13) and (4.14)

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \frac{U\sqrt{(\zeta^2-l^2)}}{\zeta}, \tag{4.15}$$

or

$$\frac{dz}{dw} = \frac{w}{U}\frac{1}{\sqrt{(w^2-U^2l^2)}}. \tag{4.16}$$

Hence

$$z = \frac{1}{U}\sqrt{(w^2-U^2l^2)}.$$

It follows too from (4.15) that the fluid velocity is infinite when $\zeta = 0$, i.e. at D.

Further, by (4.16)

$$w = \frac{Ul}{\left\{1-\left(\frac{1}{U}\frac{dw}{dz}\right)^2\right\}^{1/2}} = \frac{Ul}{\left\{1-\left(\frac{q}{U}e^{-i\theta}\right)^2\right\}^{1/2}}. \tag{4.17}$$

We are specially interested in the region for which $q > U$; in this region

$$w = -iUl\left\{\left(\frac{q}{U}e^{-i\theta}\right)^{-1} + \frac{1}{2}\left(\frac{q}{U}e^{-i\theta}\right)^{-3} + \frac{3}{8}\left(\frac{q}{U}e^{-i\theta}\right)^{-5} + \cdots\right\}$$

$$= -iUl\left\{\left(\frac{q}{U}\right)^{-1}\cos\theta + \frac{1}{2}\left(\frac{q}{U}\right)^{-3}\cos 3\theta + \frac{3}{8}\left(\frac{q}{U}\right)^{-5}\cos 5\theta + \cdots\right\}$$

$$+ Ul\left\{\left(\frac{q}{U}\right)^{-1}\sin\theta + \frac{1}{2}\left(\frac{q}{U}\right)^{-3}\sin 3\theta + \frac{3}{8}\left(\frac{q}{U}\right)^{-5}\sin 5\theta + \cdots\right\},$$

so that

$$\varphi = Ul\left\{\left(\frac{q}{U}\right)^{-1}\sin\theta + \frac{1}{2}\left(\frac{q}{U}\right)^{-3}\sin 3\theta + \cdots\right\}, \tag{4.18}$$

$$\psi = -Ul\left\{\left(\frac{q}{U}\right)^{-1}\cos\theta + \frac{1}{2}\left(\frac{q}{U}\right)^{-3}\cos 3\theta + \cdots\right\}. \tag{4.19}$$

The compressible field of flow is then determined as before by replacing q/u in (4.18) and (4.19) by the appropriate functions of τ and τ_1. The sonic line can be readily determined and is a closed curve surrounding the point D.

V. The effect of a shock wave on the drag on a body moving through a compressible fluid

Introduction

When the speed of flow of a compressible fluid past a body is such that there is a region of supersonic flow around some part of the body (i.e. when the Mach number exceeds its critical value) then, in general, a shock wave emanates from the point on the body where the transition from supersonic to subsonic flow is taking place. When the fluid passes through the shock wave, it is compressed, its temperature rises and, the process being non-adiabatic, there is an increase of entropy. Hence, only part of the mechanical work transformed into heat energy in the compression is capable of being reconverted into mechanical energy and the energy "lost" in this way manifests itself as part of the drag on the body.

Soon after a shock wave has formed, boundary layer separation takes place and leads to a further increase of drag. These two cumulative effects cause the drag coefficient to rise rapidly with increase of Mach number above the critical value. It is our object here to calculate approximately, for the cases of flow past a Rankine oval and an elliptic cylinder, the increase in drag coefficient due to the presence of a shock wave. It will be shown that this increase is much smaller than the total increase which would be observed in practice, whence we conclude that it is the boundary layer separation, rather than the shock wave, which is the responsible for the most of the drag.

1. <u>Flow through a normal shock wave</u>

Let us consider a normal shock wave and calculate the change in entropy of the fluid in passing through it.

Let quantities measured on the upstream side of the shock, the downstream side of the shock and at the stagnation point be denoted by suffixes 1, 2, 0 respectively. Thus let v_1 , p_1 , ρ_1 , c_1 , M_1 represent velocity, pressure, density, velocity of sound and Mach number upstream of the shock. Since we are contemplating a shock wave emanating from a body such as a cylinder on an aerofoil, v_1 , p_1 , ρ_1 , c_1 , M_1 (measured just upstream of the shock) will be different from the corresponding values v , p , ρ , c , M in the main stream.

Throughout, the ratio of the specific heats of the fluid at constant pressure and volume will be denoted by γ andin our calculations will be taken as 1·4 (the value for air).

The basic formulae of the theory of normal shock waves which we shall require are derived in Appendix I and are as follows:

$$\frac{p_2}{p_1} = \frac{1}{\gamma+1}\left\{2\gamma M_1^2 - (\gamma-1)\right\}, \tag{1.1}$$

or

$$\frac{p_1}{p_2} = \frac{1}{\gamma+1}\left\{2\gamma M_2^2 - (\gamma-1)\right\},$$

$$M_2 = \left\{\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right\}^{1/2}, \tag{1.2}$$

$$v_1 v_2 = \frac{2}{\gamma+1}c_0^2, \tag{1.3}$$

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma-1)M_1^2+2}{(\gamma+1)M_1^2}. \tag{1.4}$$

We shall assume in what follows that conditions are perfectly adiabatic on either side of the shock wave.

It follows from (1.3) and (1.4) that

$$\frac{v_1}{v_2} = \frac{(\gamma+1)v_1^2}{(\gamma-1)v_1^2+2c_1^2},$$

whence, by (1.3)

$$(\gamma-1)v_1^2 + 2c_1^2 = 2c_0^2,$$

or

$$v_1^2 = \frac{2}{\gamma-1}(c_0^2 - c_1^2). \tag{1.5}$$

Also, by (1.3)

$$v_2^2 = \frac{2(\gamma-1)}{(\gamma+1)^2}\,\frac{c_0^4}{c_0^2-c_1^2}. \tag{1.6}$$

Now if φ denotes the entropy per unit mass of the fluid, then the change in entropy of the fluid (per unit mass) in crossing the shock wave is given by [x]

$$\delta\varphi = C_v \log\left(\frac{p_2 \rho_1^{\gamma}}{p_1 \rho_2^{\gamma}}\right),$$

where C_v is the specific heat of the fluid at constant volume.

Hence, by (1.1) and (1.4)

$$\delta\varphi = C_v \log\left[\frac{1}{\gamma+1}\left\{2\gamma m_1^2 - (\gamma-1)\right\}\left\{\frac{(\gamma-1)m_1^2+2}{(\gamma+1)m_1^2}\right\}^{\gamma}\right],$$

i.e.

$$\delta\varphi = C_v \log\left[\frac{2\gamma v_1^2 - (\gamma-1)c_1^2}{(\gamma+1)c_1^2}\left\{\frac{(\gamma-1)v_1^2+2c_1^2}{(\gamma+1)v_1^2}\right\}^{\gamma}\right].$$

Using (1.5), this expression may be written

$$\delta\varphi = C_v \log\left[\frac{4\gamma c_0^2 - (\gamma+1)^2 c_1^2}{(\gamma^2-1)c_1^2}\cdot\left\{\frac{(\gamma-1)c_0^2}{(\gamma+1)(c_0^2-c_1^2)}\right\}^{\gamma}\right]. \qquad (1.7)$$

Now the amount of energy rendered unavailable through the irreversible nature of the compression shock is the increase of entropy multiplied by the lowest available temperature; this temperature is the temperature t of the free stream. Hence

energy rendered unavailable per unit length of shock wave $= \rho_1 v_1 t\,\delta\varphi$,

the shock wave being normal to the direction of flow. The quantities ρ_1, v_1, c_1 vary from point to point along the shock wave, so that

total energy rendered unavailable $= \int \rho_1 v_1 t\,\delta\varphi\, ds$,

where $\delta\varphi$ is given by (1.7) and the integration is

x See any standard textbook on Thermodynamics taken along the total length s of the shock wave.

Hence, if δD denotes the increase in drag due to the shock wave and δC_D the corresponding increment in the drag coefficient, then

$$v\,\delta D = \int \rho_1 v_1 t\, \delta\varphi\, ds$$

and
$$\delta C_D = \frac{\delta D}{\frac{1}{2}\rho v^2} = \frac{2}{\rho v^3}\int \rho_1 v_1 t\, \delta\varphi\, ds. \tag{1.8}$$

Since the flow round an aerofoil is determined experimentally by means of pressure measurements, it is convenient to express (1.8) wholly in terms of quantities relating to the free air stream and the pressure coefficient $\delta p \equiv \dfrac{p_1 - p}{\frac{1}{2}\rho v^2}$.

We have
$$\left(\frac{c_1}{c_0}\right)^2 = \frac{p_1}{\rho_1}\frac{\rho_0}{p_0} = \frac{p_1}{p}\frac{\rho}{\rho_1}\frac{p}{p_0}\frac{\rho_0}{\rho},$$

or, since
$$p_1\rho_1^{-\gamma} = p\rho^{-\gamma} = p_0\rho_0^{-\gamma},$$

$$\left(\frac{c_1}{c_0}\right)^2 = \left(\frac{p_1}{p}\right)^{1-\frac{1}{\gamma}}\left(\frac{p}{p_0}\right)^{1-\frac{1}{\gamma}}$$

$$= \left(\frac{p_1}{p}\right)^{1-\frac{1}{\gamma}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}, \tag{1.9}$$

by Bernoulli's Theorem.

Now $\delta p \equiv (p_1 - p)/\frac{1}{2}\rho v^2$, so that

$$\frac{p_1}{p} = 1 + \frac{1}{2}\frac{\rho v^2 \delta p}{p} = 1 + \frac{1}{2}\gamma M^2 \delta p. \tag{1.10}$$

Hence, by (1.9) and (1.10)

$$\left(\frac{c_1}{c_0}\right)^2 = \left(1 + \frac{1}{2}\gamma M^2\delta p\right)^{1-\frac{1}{\gamma}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}. \tag{1.11}$$

Thus, by (1.7), $\delta\varphi$ can be expressed entirely in terms of M and δp.

Also, by (1.5),

$$v_1^2 = \frac{2c_0^2}{\gamma-1}\left(1-\frac{c_1^2}{c_0^2}\right)$$

$$= \frac{2}{\gamma-1}\left(1-\frac{c_1^2}{c_0^2}\right)\frac{v^2}{m^2}\left(1+\frac{\gamma-1}{2}m^2\right), \qquad (1.12)$$

since $\dfrac{c_0^2}{c^2} = 1+\dfrac{\gamma-1}{2}m^2.$

Further,

$$\rho_1 = \rho\frac{\rho_1}{\rho} = \rho\left(\frac{p_1}{p}\right)^{\frac{1}{\gamma}} = \rho\left(1+\tfrac{1}{2}\gamma m^2\delta p\right)^{\frac{1}{\gamma}}, \qquad (1.13)$$

by (1.10).

Finally,

$$t = \frac{p}{R\rho} = \frac{a^2}{\gamma R} = \frac{v^2}{m^2\gamma R}, \qquad (1.14)$$

where R is the gas constant per unit mass of gas.

Thus, by (1.8), (1.7), (1.11), (1.12), (1.13), (1.14),

$$\delta C_D = \int \frac{2}{\rho v^3}\cdot\rho\left(1+\tfrac{1}{2}\gamma m^2\delta p\right)^{\frac{1}{\gamma}}\frac{v}{m}\sqrt{\frac{2}{\gamma-1}}\left(1-\frac{c_1^2}{c_0^2}\right)^{\frac{1}{2}}\left(1+\frac{\gamma-1}{2}m^2\right)^{\frac{1}{2}}\frac{v^2}{m^2\gamma R}$$

$$\times C_v \log\left[\frac{4\gamma c_0^2-(\gamma+1)^2c_1^2}{(\gamma^2-1)c_1^2}\cdot\left\{\frac{(\gamma-1)c_0^2}{(\gamma+1)(c_0^2-c_1^2)}\right\}^{\gamma}\right]ds,$$

i.e.

$$\delta C_D = \frac{2\sqrt{2}\,C_v}{m^3\gamma R(\gamma-1)^{1/2}}\int\left(1+\tfrac{1}{2}\gamma m^2\delta p\right)^{\frac{1}{\gamma}}\left(1+\frac{\gamma-1}{2}m^2\right)^{\frac{1}{2}}\left(1-\frac{c_1^2}{c_0^2}\right)^{\frac{1}{2}}$$

$$\times \log\left[\frac{4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}}{(\gamma^2-1)\frac{c_1^2}{c_0^2}}\left\{\frac{\gamma-1}{(\gamma+1)\left(1-\frac{c_1^2}{c_0^2}\right)}\right\}^{\gamma}\right]ds.$$

Since $C_v/R = (\gamma-1)^{-1}$, this may be written

$$\delta C_D = \frac{2\sqrt{2}\left(1+\frac{\gamma-1}{2}m^2\right)^{\frac{1}{2}}}{\gamma(\gamma-1)^{3/2}m^3}\int\left(1+\tfrac{1}{2}\gamma m^2\delta p\right)^{\frac{1}{\gamma}}\left(1-\frac{c_1^2}{c_0^2}\right)^{\frac{1}{2}}$$

$$\times \log\left[\frac{4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}}{(\gamma^2-1)\frac{c_1^2}{c_0^2}}\left\{\frac{\gamma-1}{(\gamma+1)\left(1-\frac{c_1^2}{c_0^2}\right)}\right\}^{\gamma}\right]ds, \qquad (1.15)$$

where c_1/c_0 is given by (1.11).

Thus (1.11) and (1.15) enable us to calculate δC_D given the Mach number M of the free stream and the pressure coefficient δp at each point of the shock wave.

We now proceed to apply these formulae to the case of flow at zero incidence past (i) a Rankine Oval and (ii) an elliptic cylinder, and shall make several simplifying assumptions. The numerical values obtained for the increase in drag coefficient due to the shock wave will therefore not be accurate but will certainly be of the correct order of magnitude; this is all that we can aim at attaining in a problem so complicated as this one.

We shall assume that the shock wave springs from the point on the body at which the pressure, in the incompressible flow, is minimum and that the shock extends along the equipotential curve from this point to the point in the fluid at which the integrand in (1.15) is zero; this latter point is the point at which the Mach number is unity. Further, we shall neglect the gradual rearward movement of the shock wave with the increase of Mach number. The values of the pressure in the compressible flow will be obtained from the corresponding values in the incompressible flow by means of the Linear

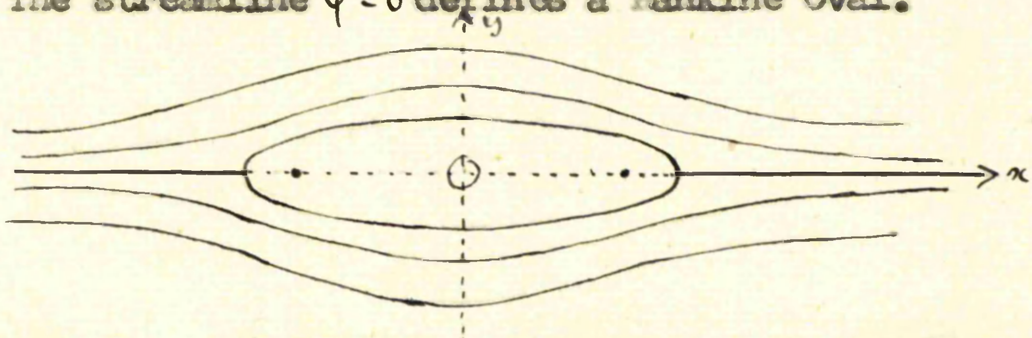
Perturbation Theory. This latter approximation is a rather crude one at high Mach numbers, but the error will be kept at a minimum by considering only slender bodies (10% thick).

2. <u>Flow past a Rankine Oval</u>

We first of all consider the incompressible flow. Consider a source and a sink of strength m situated at the points $(s, 0)$ and $(-s, 0)$ and let a uniform rectilinear motion of velocity $-U$ in the direction of the x-axis be superimposed on the motion due to the source and the sink. Then the stream function ψ for the resultant motion is given by *

$$\psi = -Uy + \frac{m}{2\pi} \tan^{-1} \frac{2ys}{x^2+y^2-s^2}.$$

The streamline $\psi = 0$ defines a Rankine Oval.



Let u, v be the x, y components of the fluid velocity at any point in the field of flow. Then

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

In the case of the compressible flow past the Rankine oval the shock wave will emanate, in accordance with our assumptions above, from the ends of the minor axis

* Glauert: Aerofoil and Airscrew Theory, Chapter III

of the oval and will extend for a certain distance along the prolongation of the minor axis. We are therefore especially interested in the fluid velocity at pointson the minor axis produced. Here $v = 0$, and

$$u = -\mathcal{U} - \frac{ms}{y^2+s^2}.$$

If we now let $y = ls$, where l is a dimensionless constant

$$u = -\mathcal{U}\left[1 + \frac{1}{1+l^2}\,\frac{m}{\mathcal{U}s}\right].$$

It is shown in Glauert (loc. cit.) that the shape of the oval depends solely on $m/\mathcal{U}s$; and if a, b denote the lengths of the semi-major and semi-minor axes respectively, then the following numerical values for a certain particular case can be easily calculated:

$\frac{\mathcal{U}b}{m}$	$\frac{\mathcal{U}s}{m}$	$\frac{a}{s}$	$\frac{b}{s}$	$\frac{a}{b}$
0•45	4•498	1•035	1/9•995	10•349 .

Thus, if we take $s = 10b$ and $\mathcal{U}s/m = 4{\cdot}5$ throughout, then our calculations will apply to an oval approximately 10% thick. Hence,

$$u = -\mathcal{U}\left[1 + \frac{1}{4{\cdot}5(1+l^2)}\right]. \tag{2.1}$$

Now

$$\delta p \equiv (p_1 - p)\Big/\tfrac{1}{2}\rho\mathcal{U}^2,$$

and, for incompressible flow

$$p + \tfrac{1}{2}\rho\mathcal{U}^2 = p_1 + \tfrac{1}{2}\rho u^2,$$

hence

$$\delta p_i = 1 - \left(\frac{u}{\mathcal{U}}\right)^2,$$

i.e.

$$\delta p_i = 1 - \left\{1 + \frac{1}{4{\cdot}5(1+l^2)}\right\}^2, \tag{2.2}$$

the subscript i being used to indicate that the formula applies to incompressible flow. The pressure coefficient δp for the compressible flow corresponding to the δp_i of the last equation is given, in accordance with the Linear Perturbation Theory, by

$$\delta p = \delta p_i / \sqrt{(1-M^2)}, \tag{2.3}$$

where M is the Mach number of the free stream (cf. (2.2) of Chapter II)).

The procedure for calculating the increase in drag coefficient of the Rankine oval due to the shock wave is then as follows:

By means of (2.2) and (2.3) we calculate δp for some fixed value of M and for various values of l. The quantity c_1/c_0 is then evaluated by means of (1.11) and δC_D calculated by (1.15). The integration has, of course, to be performed numerically and, in the case of the calculations given below, Simpson's Rule was used.

The calculations have been carried out (for a Rankine Oval 10% thick) for Mach numbers 0·90 and 0·85. The details are given in Appendix II but the results are as follows

M	0·90	0·85	0·72
δC_D	0·030	0·024	0 .

δC_D is, of course, zero at the critical Mach number of the oval. $\left(\frac{u}{U}\right)$ max. has the value $11/9$ by (2.1) whence the value 0·72 for the critical Mach number is obtained from the linear perturbation theory.

3. Flow past an elliptic cylinder *

Here again we begin by considering the incompressible flow and determine it from the flow past a circular cylinder by the method of conformal representation.

The transformation $z' = z + \frac{a^2}{z}$,

applied to the flow past the cylinder $|z| = b$ $(b>a)$ in the z - plane gives the flow past an elliptic cylinder in the z' - plane. For, since

$$z' \pm 2a = (z \pm a)^2/z$$

then, if P, A., B are the points z, $+a$, $-a$ respectively and O is the origin,

$$\begin{aligned} |z'+2a| + |z'-2a| &= (AP^2 + BP^2)/b \\ &= 2(OA^2 + OP^2)/b \\ &= 2(a^2+b^2)/b. \end{aligned}$$

Hence, the point P', which represents z', describes an ellipse whose foci are at the points $z' = \pm 2a$. The major and minor axes of the

* The method followed here is that given by S.L. Green: Hydro - and Aerodynamics

ellipse are of lengths $2(a^2+b^2)/b$ and $2(b^2-a^2)/b$ respectively.

If β is the angle of incidence of the flow with the x-axis, $-U$ is the velocity at infinity and K is the circulation, then the motion in the z'- plane is determined by the complex potential function $w \equiv \varphi + i\psi$ where

$$w = U(ze^{-i\beta} + b^2e^{i\beta}/z) + \frac{iK}{2\pi}\log\frac{z}{a}$$

and

$$z' = z + \frac{a^2}{z}.$$

We take $K = \beta = 0$, so that

$$w = U(z + \frac{b^2}{z}) \tag{3.1}$$

and

$$z' = z + \frac{a^2}{z}. \tag{3.2}$$

Now let $z = ae^{\zeta} = ae^{\xi+i\eta} = be^{\zeta-\alpha}$,

where $e^{\alpha} = b/a$. Then

$$w = Ub(e^{\zeta-\alpha} + e^{\alpha-\zeta}) = 2Ub\cosh(\zeta-\alpha), \tag{3.3}$$

and

$$z' = 2a\cosh\zeta. \tag{3.4}$$

When z is on the circle $|z| = b$ and therefore z' is on the ellipse, $\zeta-\alpha$ is purely imaginary. Hence ξ is const. ($=\alpha$) and η may be assumed to vary from 0 to 2π.

The velocity (u', v') at the point z' in the z' - plane is given by

$$-u' + iv' = \frac{dw}{dz'} = \frac{dw}{d\zeta}\frac{d\zeta}{dz'}$$

$$= 2Ub\sinh(\zeta-\alpha)/2a\sinh\zeta \quad \text{by (3.3), (3.4)}$$

$$= \frac{Ub}{a}\sinh(\zeta-\alpha)\operatorname{cosech}\zeta. \tag{3.5}$$

On the cylider, $\zeta = \alpha + i\eta$, whence by the last equation

$$-u' + iv' = \frac{Ub}{a} \sinh i\eta \operatorname{cosech}(\alpha + i\eta),$$

or
$$= \frac{iUb}{a} \sin\eta \operatorname{cosech}(\alpha + i\eta). \tag{3.6}$$

As in the case of the Rankine oval, we are especially interested in the velocity at points on the prolongation of the minor axis for it is along that line that the shock wave is assumed to extend. At points on the minor axis,

$$z' = iy'$$

and
$$2a \cosh\zeta = iy',$$

i.e.
$$e^{2\zeta} - \frac{iy'}{a} e^{\zeta} + 1 = 0.$$

Hence
$$e^{\zeta} = \frac{i}{2a}\left\{y' \pm \sqrt{(y'^2 + 4a^2)}\right\}$$

and
$$e^{-\zeta} = \frac{i}{2a}\left\{y' \mp \sqrt{(y'^2 + 4a^2)}\right\}.$$

Thus,
$$\sinh\zeta = \pm \frac{i}{2a}\sqrt{(y'^2 + 4a^2)},$$

and
$$\sinh(\zeta - \alpha) = \sinh\zeta \cosh\alpha + \cosh\zeta \sinh\alpha$$
$$= \pm \frac{i}{2a}\sqrt{(y'^2 + 4a^2)}\cosh\alpha + \frac{i}{2a} y' \sinh\alpha.$$

Hence, by (3.5)
$$-u' + iv' = \frac{Ub}{a}\left\{\cosh\alpha \pm \frac{y'}{\sqrt{(y'^2 + 4a^2)}}\sinh\alpha\right\}$$

and, since $v' = 0$ at the points we are considering,

$$u' = -\frac{Ub}{a}\left[\frac{1}{2}\frac{a^2 + b^2}{ab} \pm \frac{y'}{\sqrt{(y'^2 + 4a^2)}}\frac{1}{2}\frac{b^2 - a^2}{ab}\right],$$

i.e.
$$u' = -U\left[a^2 + b^2 \pm \frac{y'}{\sqrt{(y'^2 + 4a^2)}}(b^2 - a^2)\right]\Big/2a^2. \tag{3.7}$$

To obtain the velocity distribution along the prolongation of the minor axis of the elliptic cylinder

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1,$$

we put $p = (a^2 + b^2)/b$, $q = (b^2 - a^2)/b$. We then have

$$\frac{b^2}{a^2} = \frac{p+q}{p-q}, \qquad 4a^2 = p^2 - q^2,$$

and (3.7) becomes

$$u' = -\frac{U}{2}\left(1 + \frac{p+q}{p-q}\right)\left\{1 \pm \frac{q}{p}\frac{y'}{\sqrt{(y'^2 + p^2 - q^2)}}\right\}.$$

To ensure that $u' = -U$ when $y' = \pm\infty$, we require to take the negative sign when y' is positive and the positive sign when y' is negative; the flow is perfectly symmetrical. Thus, for positive values of y', we have

$$u' = -\tfrac{1}{2}U\left(1 + \frac{p+q}{p-q}\right)\left\{1 - \frac{q}{p}\frac{y'}{\sqrt{(y'^2 + p^2 - q^2)}}\right\}. \tag{3.8}$$

Exactly as in the case of the Rankine oval, we then have

$$\delta p = \delta p_i / \sqrt{(1 - m^2)},$$

where

$$\delta p_i = 1 - \left(\frac{u'}{U}\right)^2$$

and u'/U is given by (3.8).

The calculations, which are given in detail in the Appendix, were carried out for an elliptic cylinder 10% thick, i.e. $p = 10q$, and for Mach numbers 0•95 and 0·90. In this case (3.8) reduces to

$$\left|\frac{u'}{U}\right| = \frac{10}{9}\left\{1 - \frac{\ell}{\sqrt{(100\,\ell^2 + 99)}}\right\}, \tag{3.9}$$

where $y' = \ell p$.

The results obtained are as follows:

M	0·95	0·90	0·83
δC_D	0·012	0·0010	0

The critical Mach number is calculated as in the case of the Rankine oval. $\left(\frac{u'}{U}\right)$ max is approximately $10/9$ (since $\ell = 1/10$ for the end of the minor axis) and the critical Mach number turns out to be 0·83.

4. Conclusions

Apart from the fact that the increase of drag coefficient is greater for the Rankine oval than for the elliptic cylinder, two conclusions can be drawn from the results of our calculations.

Firstly, the increase of drag coefficient rises rapidly with the Mach number and, in the case of the elliptic cylinder, would seem to depend on the 3rd or 4th power of the excess of the Mach number over its critical value.

Secondly, the increase of drag coefficient due to the shock wave is only a very small component of the total increase as observed in practice. That this is so can be seen in the following way. It has been observed experimentally that when the Mach number exceeds its critical value by 0·1, the observed increase in the drag coefficient of an aerofoil 9% thick is of the order of 0·03 or 0·04 *. The corresponding figure for an elliptic cylinder will be many times as large (of the order of 10 times at least), whence, since the theoretical figure is about 0·0045, we see that the increase in drag coefficient is very largely due to the boundary layer separation caused by the shock wave rather than to the shock wave itself. Even if the increase of drag coefficient due to the shock wave was as great for the aerofoil as for the elliptic cylinder, then the observed increase of drag coefficient of aerofoil would be about 10 times the calculated value.

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* I am indebted to Prof. A. Thom for this information. It was given to me in 1945 and I think the source was an American report.

Appendix I. Derivation of Formulae

(1.1) (1.2) (1.3) (1.4)

The fundamental formulae of the theory of normal shock waves are usually stated in the following way*

$$\rho_1 v_1 = \rho_2 v_2 , \qquad (1) \text{ (conservation of mass)}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 , \qquad (2) \text{ (conservation of momentum)}$$

$$i_1 + \tfrac{1}{2} v_1^2 = i_2 + \tfrac{1}{2} v_2^2 . \qquad (3) \text{ (conservation of energy)}$$

In the last equation, i denotes the total heat or enthalpy and we note that

$$i = c_p t = \frac{c^2}{\gamma - 1} \qquad (4)$$

since $c_p = R\gamma/(\gamma-1)$.

From (2) we have immediately that

$$p_1(1+\gamma M_1^2) = p_2(1+\gamma M_2^2) ,$$

or
$$\frac{p_1}{p_2} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2} . \qquad (5)$$

Also, by (1)

$$\frac{v_1 p_1}{c_1^2} = \frac{v_2 p_2}{c_2^2} , \qquad (6)$$

i.e.
$$\frac{M_1 p_1}{c_1} = \frac{M_2 p_2}{c_2}$$

and therefore, by (6)

$$\frac{c_1}{c_2} = \frac{M_1}{M_2} \frac{1+\gamma M_2^2}{1+\gamma M_1^2} . \qquad (7)$$

Thus

$$\frac{v_1}{v_2} = \frac{M_1}{M_2}\frac{c_1}{c_2} = \frac{M_1^2}{M_2^2}\frac{1+\gamma M_2^2}{1+\gamma M_1^2} . \qquad (8)$$

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* See, for example Busemann: Handbuch der Experimentalphysik, Vol. 4 (1931) § 27

Further, by (3) and (4)

$$\frac{c_1^2}{\gamma-1} + \frac{1}{2}c_1^2 M_1^2 = \frac{c_2^2}{\gamma-1} + \frac{1}{2}c_2^2 M_2^2,$$

whence, by (7)

$$\frac{M_1^2}{M_2^2}\left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)^2 = \frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}.$$

On simplification, this last equation reduces to (1.2),

i.e. $$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}. \qquad (1.2)$$

Substituting for M_2 in terms of M_1 or vice-versa in (5), we immediately obtain (1.1)

i.e. $$\frac{p_2}{p_1} = \frac{1}{\gamma+1}\{2\gamma M_1^2-(\gamma-1)\} \qquad (1.1)$$

or $$\frac{p_1}{p_2} = \frac{1}{\gamma+1}\{2\gamma M_2^2-(\gamma-1)\}.$$

Similarly (1.2) and (8) give (1.4)

i.e. $$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma-1)M_1^2+2}{(\gamma+1)M_1^2}. \qquad (1.4)$$

We do not derive (1.3) here for it can be found in any standard text book.

These formulae which we have derived as alternatives to the standard formulae (1), (2) and (3) are not original.

While at the Royal Aircraft Establishment, I saw them quoted without proof in an unpublished report * which happened to come my way.

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* Thompson: A summary of Formulae and Table s for calculations of compressible air flow. Tech. Note No. Aero. 1613 (1945).

Appendix II - Calculation of δC_D

The procedure adopted was as follows. The value of M having been decided upon, δp was calculated for various values of l by means of (2.2) and (2.3) in the case of the Rankine oval and by the corresponding equations in the case of the elliptic cylinder. c_1/c_0 was then evaluated by means of (1.11) whence δC_D was determined from (1.15). The numerical integration was carried out by Simpson's Rule. It was found in the cases considered here that the integrand in (1.15) is nearly zero when $l = 1$ whence, to the degree of approximation used elsewhere in our work, it was unnecessary to consider values of l greater than 1. The calculations were carried out by 4-figure tables but in certain places where the fourth figure is very significant its value was checked by 5-figure tables.

For brevity, we let

$$\log\left[\frac{4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}}{(\gamma^2-1)\frac{c_1^2}{c_0^2}}\cdot\left\{\frac{\gamma-1}{(\gamma+1)\left[1-\frac{c_1^2}{c_0^2}\right]}\right\}^\gamma\right] = \log\left\{4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}\right\}+\gamma\log\frac{\gamma-1}{\gamma+1}$$

$$-\log(\gamma^2-1)-\log\frac{c_1^2}{c_0^2}-\gamma\log\left(1-\frac{c_1^2}{c_0^2}\right)$$

$$\equiv A-B,$$

and

$$C \equiv \log\frac{2.303\,.\,2\sqrt{2}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}}}{\gamma(\gamma-1)^{3/2}M^3}.$$

Rankine Oval. M = 0·9

l	0·1	0·2	0·3	0·4	0·5
$1+l^2$	1·01	1·04	1·09	1·16	1·25
$\log(1+l^2)$	0·0043	0·0170	0·0374	0·0645	0·0969
$\log 4\cdot5$	0·6532	0·6532	0·6532	0·6532	0·6532
	0·6575	0·6702	0·6906	0·7177	0·7501
	$\bar{1}$·3425	$\bar{1}$·3298	$\bar{1}$·3094	$\bar{1}$·2823	$\bar{1}$·2499
u/U	1·2201	1·2137	1·2039	1·1913	1·1778
$\log u/U$	0·0864	0·0841	0·0806	0·0761	0·0711
$2\log u/U$	0·1728	0·1682	0·1612	0·1522	0·1422
$(u/U)^2$	1·489	1·473	1·450	1·420	1·388
$-\delta p_i$	0·489	0·473	0·450	0·420	0·388
$-\delta p_i/\sqrt{(1-m^2)}$	0·850	0·822	0·782	0·730	0·674
$-\frac{1}{2}\gamma m^2\delta p$	0·482	0·466	0·443	0·414	0·382
$\log(1+\frac{1}{2}\gamma m^2\delta p)$	$\bar{1}$·7143	$\bar{1}$·7275	$\bar{1}$·7459	$\bar{1}$·7679	$\bar{1}$·7910
$\log(1+\frac{1}{2}\gamma m^2\delta p)^{1-\frac{1}{\gamma}}$	$\bar{1}$·9184	$\bar{1}$·9221	$\bar{1}$·9274	$\bar{1}$·9337	$\bar{1}$·9403
$\log(1+\frac{\gamma-1}{2}m^2)$	0·0653	0·0653	0·0653	0·0653	0·0653
$\log\left(\frac{c_1}{c_0}\right)^2$	$\bar{1}$·8531	$\bar{1}$·8568	$\bar{1}$·8621	$\bar{1}$·8684	$\bar{1}$·8750
$\left(\frac{c_1}{c_0}\right)^2$	0·7131	0·7191	0·7280	0·7386	0·7490
$1-\left(\frac{c_1}{c_0}\right)^2$	0·2869	0·2809	0·2720	0·2614	0·2501
$\log\left[1-\left(\frac{c_1}{c_0}\right)^2\right]$	$\bar{1}$·4578	$\bar{1}$·4486	$\bar{1}$·4346	$\bar{1}$·4173	$\bar{1}$·3981
$\log\left[1-\left(\frac{c_1}{c_0}\right)^2\right]^{1/2}$	$\bar{1}$·7289	$\bar{1}$·7243	$\bar{1}$·7173	$\bar{1}$·7086	$\bar{1}$·6990

l	0·6	0·7	0·8	0·9	1·0
$1+l^2$	1·36	1·49	1·64	1·21	2·0
$\log(1+l^2)$	0·1335	0·1732	0·2148	0·2553	0·3010
$\log 4{\cdot}5$	0·6532	0·6532	0·6532	0·6532	0·6532
	0·7867	0·8264	0·8680	0·9085	0·9542
	$\bar{1}$·2133	$\bar{1}$·1736	$\bar{1}$·1320	$\bar{1}$·0915	$\bar{1}$·0458
u/U	1·1634	1·1491	1·1355	1·1234	1·1111
$\log(u/U)$	0·0658	0·0607	0·0552	0·0505	0·0457
$2\log(u/U)$	0·1316	0·1214	0·1104	0·1010	0·0914
$(u/U)^2$	1·354	1·322	1·289	1·262	1·234
$-\delta p_i$	0·354	0·322	0·289	0·262	0·234
$-\delta p_i/\sqrt{1-m^2}$	0·615	0·560	0·502	0·455	0·407
$-\frac{1}{2}\gamma m^2\delta p$	0·349	0·318	0·285	0·258	0·231
$\log(1+\frac{1}{2}\gamma m^2\delta p)$	$\bar{1}$·8136	$\bar{1}$·8338	$\bar{1}$·8543	$\bar{1}$·8704	$\bar{1}$·8859
$\log(1+\frac{1}{2}\gamma m^2\delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9467	$\bar{1}$·9525	$\bar{1}$·9584	$\bar{1}$·9630	$\bar{1}$·9674
$\log(1+\frac{\gamma-1}{2}m^2)$	0·0653	0·0653	0·0653	0·0653	0·0653
$\log\left(\frac{c_1}{c_0}\right)^2$	$\bar{1}$·8814	$\bar{1}$·8872	$\bar{1}$·8931	$\bar{1}$·8977	$\bar{1}$·9021
$\left(\frac{c_1}{c_0}\right)^2$	0·7610	0·7713	0·7818	0·7902	0·7982
$1-\left(\frac{c_1}{c_0}\right)^2$	0·2390	0·2287	0·2182	0·2098	0·2018
$\log\left[1-\left(\frac{c_1}{c_0}\right)^2\right]$	$\bar{1}$·3784	$\bar{1}$·3583	$\bar{1}$·3389	$\bar{1}$·3218	$\bar{1}$·3049
$\log\left[1-\left(\frac{c_1}{c_0}\right)^2\right]^{1/2}$	$\bar{1}$·6892	$\bar{1}$·6796	$\bar{1}$·6694	$\bar{1}$·6609	$\bar{1}$·6525

$(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	4·107	4·142	4·193	4·255	4·319
$4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	1·493	1·458	1·407	1·345	1·281
$\log\left[4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}\right]$	0·1741	0·1638	0·1482	0·1289	0·1075
$\gamma \log \frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{1}$·0846	$\bar{1}$·0743	$\bar{1}$·0587	$\bar{1}$·0394	$\bar{1}$·0180
$\gamma \log\left(1-\frac{c_1^2}{c_0^2}\right)$	$\bar{1}$·2409	$\bar{1}$·2281	$\bar{1}$·2084	$\bar{1}$·1841	$\bar{1}$·1573
$\log c_1^2/c_0^2$	$\bar{1}$·8531	$\bar{1}$·8562	$\bar{1}$·8621	$\bar{1}$·8684	$\bar{1}$·8750
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{1}$·0763	$\bar{1}$·0672	$\bar{1}$·0528	$\bar{1}$·0348	$\bar{1}$·0146
A - B	0·0083	0·0071	0·0059	0·0046	0·0034
C	1·4341	1·4341	1·4341	1·4341	1·4341
$\log\left[1-\frac{c_1^2}{c_0^2}\right]^{1/2}$	$\bar{1}$·7289	$\bar{1}$·7243	$\bar{1}$·7173	$\bar{1}$·7086	$\bar{1}$·6990
$\log\left[1+\frac{1}{2}\gamma M^2 \delta p\right]^{1/\gamma}$	$\bar{1}$·7959	$\bar{1}$·8054	$\bar{1}$·8185	$\bar{1}$·8342	$\bar{1}$·8507
$\log(A-B)$	$\bar{2}$·9191	$\bar{3}$·8513	$\bar{3}$·7709	$\bar{3}$·6628	$\bar{3}$·5315
	$\bar{2}$·8780	$\bar{2}$·8151	$\bar{2}$·7408	$\bar{2}$·6397	$\bar{2}$·515[illegible]
Integrand $\times 10^2$	7.551	6·653	5·505	4·362	3·275

Hence, by Simpson's Rule

$$10^2 \delta C_D = \frac{1}{30}\left[8\cdot 255 + 2(10\cdot 361) + 4(14\cdot 459)\right] + 0\cdot 055$$

so that

$$\delta C_D = 0\cdot 0295$$

i.e. $\delta C_D = 0\cdot 030$.

$(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	4·383	4·442	4·503	4·551	4·597
$4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	1·217	1·158	1·097	1·049	1·008
$\log\left[4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}\right]$	0·0854	0·0657	0·0405	0·0207	0·0012
$\gamma \log \frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{2}$·9959	$\bar{2}$·9742	$\bar{2}$·9510	$\bar{2}$·9312	$\bar{2}$·9117
$\gamma \log\left(1-\frac{c_1^2}{c_0^2}\right)$	$\bar{1}$·1298	$\bar{1}$·1031	$\bar{1}$·0745	$\bar{1}$·0505	$\bar{1}$·0269
$\log \frac{c_1^2}{c_0^2}$	$\bar{1}$·8814	$\bar{1}$·8892	$\bar{1}$·8931	$\bar{1}$·8977	$\bar{1}$·9021
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{2}$·9935	$\bar{2}$·9726	$\bar{2}$·9499	$\bar{2}$·9305	$\bar{2}$·9113
A − B	0·0024	0·0016	0·0011	0·0007	0·0004
C	1·4341	1·4341	1·4341	1·4341	1·4341
$\log\left[1-\frac{c_1^2}{c_0^2}\right]^{1/2}$	$\bar{1}$·6892	$\bar{1}$·6796	$\bar{1}$·6694	$\bar{1}$·6609	$\bar{1}$·6525
$\log\left[1+\frac{1}{2}\gamma M^2 \delta p\right]^{1/2}$	$\bar{1}$·8669	$\bar{1}$·8813	$\bar{1}$·8959	$\bar{1}$·9074	$\bar{1}$·9185
$\log(A-B)$	$\bar{3}$·3802	$\bar{3}$·2041	$\bar{3}$·0414	$\bar{4}$·8451	$\bar{4}$·6021
	$\bar{2}$·3704	$\bar{2}$·1991	$\bar{2}$·0408	$\bar{3}$·8475	$\bar{3}$·6071
Integrand. 10^2	2·346	1·581	1·098	0·7039	0·4047

Rankine oval. M = 0·85

l	0·1	0·2	0·3	0·4	0·5
$-\delta p_i$	0·489	0·473	0·450	0·420	0·388
$\log(-\delta p_i)$	$\bar{1}$·6893	$\bar{1}$·6749	$\bar{1}$·6532	$\bar{1}$·6232	$\bar{1}$·5888
$\log(-\frac{1}{2}\gamma M^2 \delta p)$	$\bar{1}$·6716	$\bar{1}$·6572	$\bar{1}$·6355	$\bar{1}$·6055	$\bar{1}$·5711
$-\frac{1}{2}\gamma M^2 \delta p$	0·4695	0·4541	0·4320	0·4032	0·3725
$\log(1+\frac{1}{2}\gamma M^2 \delta p)$	$\bar{1}$·7247	$\bar{1}$·7371	$\bar{1}$·7543	$\bar{1}$·7758	$\bar{1}$·7976
$\log(1+\frac{1}{2}\gamma M^2 \delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9214	$\bar{1}$·9249	$\bar{1}$·9298	$\bar{1}$·9359	$\bar{1}$·9422
$\log(1+\frac{\gamma-1}{2}M^2)$	0·0587	0·0587	0·0587	0·0587	0·0587
$\log c_1^2/c_0^2$	$\bar{1}$·8627	$\bar{1}$·8663	$\bar{1}$·8711	$\bar{1}$·8772	$\bar{1}$·8835
c_1^2/c_0^2	0·7290	0·7350	0·7432	0·7537	0·7647
$1-c_1^2/c_0^2$	0·2710	0·2650	0·2568	0·2463	0·2353
$\log(1-c_1^2/c_0^2)$	$\bar{1}$·4330	$\bar{1}$·4232	$\bar{1}$·4096	$\bar{1}$·3914	$\bar{1}$·3717
$(\gamma+1)^2\frac{c_1^2}{c_0^2}$	4·199	4·234	4·281	4·341	4·404
$4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}$	1·401	1·366	1·319	1·259	1·196
$\log\left[4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}\right]$	0·1464	0·1354	0·1202	0·1000	0·0778
$\gamma\log\frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{1}$·0569	$\bar{1}$·0459	$\bar{1}$·0307	$\bar{1}$·0105	$\bar{2}$·9883
$\gamma\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·2062	$\bar{1}$·1925	$\bar{1}$·1732	$\bar{1}$·1475	$\bar{1}$·1198
$\log\frac{c_1^2}{c_0^2}$	$\bar{1}$·8627	$\bar{1}$·8663	$\bar{1}$·8711	$\bar{1}$·8772	$\bar{1}$·8835
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{1}$·0512	$\bar{1}$·0411	$\bar{1}$·0266	$\bar{1}$·0070	$\bar{2}$·9856

l	0·6	0·7	0·8	0·9	1·0
$-\delta p_i$	0·354	0·322	0·289	0·262	0·234
$\log(-\delta p_i)$	$\bar{1}$·5490	$\bar{1}$·5079	$\bar{1}$·4609	$\bar{1}$·4183	$\bar{1}$·3692
$\log(-\frac{1}{2}\gamma m^2 \delta p)$	$\bar{1}$·5313	$\bar{1}$·4902	$\bar{1}$·4432	$\bar{1}$·4006	$\bar{1}$·3515
$-\frac{1}{2}\gamma m^2 \delta p$	0·3398	0·3091	0·2774	0·2516	0·2247
$\log(1+\frac{1}{2}\gamma m^2 \delta p)$	$\bar{1}$·8196	$\bar{1}$·8394	$\bar{1}$·8589	$\bar{1}$·8741	$\bar{1}$·8895
$\log(1+\frac{1}{2}\gamma m^2 \delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9485	$\bar{1}$·9541	$\bar{1}$·9597	$\bar{1}$·9640	$\bar{1}$·9684
$\log(1+\frac{\gamma-1}{2} m^2)$	0·0587	0·0587	0·0587	0·0587	0·0587
$\log c_1^2/c_0^2$	$\bar{1}$·8898	$\bar{1}$·8954	$\bar{1}$·9011	$\bar{1}$·9053	$\bar{1}$·9098
c_1^2/c_0^2	0·7759	0·7859	0·7964	0·8041	0·8125
$1-c_1^2/c_0^2$	0·2241	0·2141	0·2036	0·1959	0·1875
$\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·3504	$\bar{1}$·3306	$\bar{1}$·3088	$\bar{1}$·2920	$\bar{1}$·2730
$(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	4·469	4·527	4·586	4·631	4·679
$4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}$	1·131	1·073	1·014	0·969	0·921
$\log[4\gamma-(\gamma+1)^2 \frac{c_1^2}{c_0^2}]$	0·0535	0·0306	0·0060	$\bar{1}$·9863	$\bar{1}$·9643
$\gamma\log\frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{2}$·9640	$\bar{2}$·9411	$\bar{2}$·9165	$\bar{2}$·8968	$\bar{2}$·8748
$\gamma\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·0900	$\bar{1}$·0623	$\bar{1}$·0323	$\bar{1}$·0085	$\bar{2}$·9822
$\log\frac{c_1^2}{c_0^2}$	$\bar{1}$·8898	$\bar{1}$·8954	$\bar{1}$·9011	$\bar{1}$·9053	$\bar{1}$·9098
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{2}$·9621	$\bar{2}$·9400	$\bar{2}$·9157	$\bar{2}$·8961	$\bar{2}$·8743

$A-B$	·0057	·0048	·0041	·0035	·0027
C	1·5056	1·5056	1·5056	1·5056	1·5056
$\log\left[1-\frac{c_1^2}{c_0^2}\right]^{1/2}$	$\bar{1}\cdot7165$	$\bar{1}\cdot7116$	$\bar{1}\cdot7048$	$\bar{1}\cdot6957$	$\bar{1}\cdot6852$
$\log(A-B)$	$\bar{3}\cdot7559$	$\bar{3}\cdot6812$	$\bar{3}\cdot6128$	$\bar{3}\cdot5441$	$\bar{3}\cdot4314$
$\log\left[1+\frac{1}{2}\gamma M^2\delta p\right]^{1/2}$	$\bar{1}\cdot8034$	$\bar{1}\cdot8122$	$\bar{1}\cdot8245$	$\bar{1}\cdot8399$	$\bar{1}\cdot8554$
	$\bar{2}\cdot7814$	$\bar{2}\cdot7106$	$\bar{2}\cdot6477$	$\bar{2}\cdot5853$	$\bar{2}\cdot4782$
Integrand $\times 10^2$	6·045	5·136	4·443	3·849	3·007

Hence, by Simpson's Rule

$$10^2\,\delta C_D = \frac{1}{30}\left[6\cdot352 + 17\cdot60 + 48\cdot17\right] + \frac{1}{15}\;0\cdot69$$

and $\delta C_D = 0\cdot024$.

A - B	·0019	·0011	·0008	·0007	·0005
C	1·5056	1·5056	1·5056	1·5056	1·5056
$\log\left(1-\frac{c_1^2}{c_0^2}\right)^{1/2}$	$\bar{1}$·6752	$\bar{1}$·6653	$\bar{1}$·6544	$\bar{1}$·6459	$\bar{1}$·6365
$\log(A-B)$	$\bar{3}$·2788	$\bar{3}$·0414	$\bar{4}$·9031	$\bar{4}$·8451	$\bar{4}$·6990
$\log\left(1+\frac{1}{2}\gamma M^2\delta p\right)^{1/\gamma}$	$\bar{1}$·8711	$\bar{1}$·8853	$\bar{1}$·8992	$\bar{1}$·9101	$\bar{1}$·9211
	$\bar{2}$·3307	$\bar{2}$·0976	$\bar{2}$·9623	$\bar{3}$·9067	$\bar{3}$·7622
Integrand. 10^2	2·141	1·252	0·917	0·807	0·578

Elliptic Cylinder M = 095

l	0·1	0·2	0·3	0·4
u/U	1·1000	1·0892	1·0790	1·0697
$-(1-u^2/U^2)$	0·2100	0·1864	0·1643	0·1442
$\log(-\delta p_i)$	$\bar{1}$·3222	$\bar{1}$·2704	$\bar{1}$·2155	$\bar{1}$·1591
$\log(-\delta p)$	$\bar{1}$·8277	$\bar{1}$·7759	$\bar{1}$·7210	$\bar{1}$·6646
$\log(-\frac{1}{2}\gamma M^2\delta p)$	$\bar{1}$·6282	$\bar{1}$·5764	$\bar{1}$·5215	$\bar{1}$·4651
$-\frac{1}{2}\gamma M^2\delta p$	0·4248	0·3770	0·3323	0·2918
$\log(1+\frac{1}{2}\gamma M^2\delta p)$	$\bar{1}$·7599	$\bar{1}$·7945	$\bar{1}$·8246	$\bar{1}$·8501
$\log(1+\frac{1}{2}\gamma M^2\delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9314	$\bar{1}$·9413	$\bar{1}$·9499	$\bar{1}$·9572
$\log(1+\frac{\gamma-1}{2}M^2)$	0·0721	0·0721	0·0721	0·0721
$\log(\frac{c_1}{c_0})^2$	$\bar{1}$·8593	$\bar{1}$·8692	$\bar{1}$·8778	$\bar{1}$·8851
$(c_1/c_0)^2$	0·7233	0·7399	0·7548	0·7676
$\log(1-c_1^2/c_0^2)$	$\bar{1}$·4420	$\bar{1}$·4132	$\bar{1}$·3896	$\bar{1}$·3662
$(\gamma+1)^2 c_1^2/c_0^2$	4·166	4·262	4·347	4·421
$\log\left[4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}\right]$	0·1565	0·1265	0·0979	0·0714
$\gamma\log\frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{1}$·0670	$\bar{1}$·0370	$\bar{1}$·0084	$\bar{2}$·9819
$\gamma\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·2188	$\bar{1}$·1813	$\bar{1}$·1454	$\bar{1}$·1127
$\log\frac{c_1^2}{c_0^2}$	$\bar{1}$·8593	$\bar{1}$·8692	$\bar{1}$·8778	$\bar{1}$·8851
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{1}$·0604	$\bar{1}$·0328	$\bar{1}$·0055	$\bar{2}$·9801
A − B	0·0066	0·0042	0·0029	0·0018
C	1·3675	1·3675	1·3675	1·3675
$\log(A-B)$	$\bar{3}$·8195	$\bar{3}$·6232	$\bar{3}$·4624	$\bar{3}$·2553
$\log(1-\frac{c_1^2}{c_0^2})^{1/2}$	$\bar{1}$·7210	$\bar{1}$·7076	$\bar{1}$·6948	$\bar{1}$·6831
$\log\left[1+\frac{1}{2}\gamma M^2\delta p\right]^{1/\gamma}$	$\bar{1}$·8285	1·8532	1·8747	1·8929

	$\bar{2}\cdot7365$	$\bar{2}\cdot5515$	$\bar{2}\cdot3994$	$\bar{2}\cdot1988$
Integrand . 10^2	5·461	3·560	2·508	1·581

Hence, by Simpson's Rule

$$\delta C_D = 0\cdot012.$$

l	0·5	0·6	0·7	0·8
u/U	1·0612	1·0537	1·0472	1·0415
$-(1-u^2/U^2)$	0·1262	0·1103	0·0966	0·0847
$\log(-\delta p_i)$	$\bar{1}$·1009	$\bar{1}$·0425	$\bar{2}$·9850	$\bar{2}$·9279
$\log(-\delta p)$	$\bar{1}$·6064	$\bar{1}$·5480	$\bar{1}$·4905	$\bar{1}$·4334
$\log(-\frac{1}{2}\gamma m^2\delta p)$	$\bar{1}$·4069	$\bar{1}$·3485	$\bar{1}$·2910	$\bar{1}$·2339
$-\frac{1}{2}\gamma m^2\delta p$	0·2552	0·2231	0·1954	0·1714
$\log(1+\frac{1}{2}\gamma m^2\delta p)$	$\bar{1}$·8721	$\bar{1}$·8904	$\bar{1}$·9056	$\bar{1}$·9183
$\log(1+\frac{1}{2}\gamma m^2\delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9635	$\bar{1}$·9687	$\bar{1}$·9730	$\bar{1}$·9767
$\log(1+\frac{\gamma-1}{2}m^2)$	0·0721	0·0721	0·0721	0·0721
$\log(\frac{c_1}{c_0})^2$	$\bar{1}$·8914	$\bar{1}$·8966	$\bar{1}$·9009	$\bar{1}$·9046
$(\frac{c_1}{c_0})^2$	0·7787	0·7881	0·7960	0·8028
$\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·3450	$\bar{1}$·3261	$\bar{1}$·3096	$\bar{1}$·2949
$(\gamma+1)^2\frac{c_1^2}{c_0^2}$	4·485	4·539	4·584	4·624
$\log[4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}]$	0·0472	0·0257	0·0068	$\bar{1}$·9894
$\gamma\log\frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{2}$·9577	$\bar{2}$·9362	$\bar{2}$·9173	$\bar{2}$·8999
$\gamma\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·0830	$\bar{1}$·0566	$\bar{1}$·0336	$\bar{1}$·0128
$\log\frac{c_1^2}{c_0^2}$	$\bar{1}$·8914	$\bar{1}$·8966	$\bar{1}$·9009	$\bar{1}$·9046
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{2}$·9567	$\bar{2}$·9355	$\bar{2}$·9168	$\bar{2}$·8997
$A-B$	0·0010	0·0007	0·0005	0·0002
C	1·3675	1·3675	1·3675	1·3675
$\log(A-B)$	$\bar{3}$·0000	$\bar{4}$·8451	$\bar{4}$·6990	$\bar{4}$·3010
$\log(1-\frac{c_1^2}{c_0^2})^{1/2}$	$\bar{1}$·6725	$\bar{1}$·6631	$\bar{1}$·6548	$\bar{1}$·6475
$\log[1+\frac{1}{2}\gamma m^2\delta p]^{1/\gamma}$	$\bar{1}$·9086	$\bar{1}$·9217	$\bar{1}$·9326	$\bar{1}$·9416

	$\bar{3}\cdot9486$	$\bar{3}\cdot7974$	$\bar{3}\cdot6539$	$\bar{3}\cdot2576$
Integrand . 10^2	$0\cdot8884$	$0\cdot6772$	$0\cdot4507$	$0\cdot1809$

Elliptic Cylinder M = 0·90

l	0·1	0·2	0·3	0·4
$\log(-\delta p)$	$\bar{1}$·6828	$\bar{1}$·6310	$\bar{1}$·5761	$\bar{1}$·5197
$\log(-\frac{1}{2}\gamma M^2\delta p)$	$\bar{1}$·4364	$\bar{1}$·3846	$\bar{1}$·3297	$\bar{1}$·2733
$\log(1+\frac{1}{2}\gamma M^2\delta p)$	$\bar{1}$·8614	$\bar{1}$·8794	$\bar{1}$·8956	$\bar{1}$·9098
$\log(1+\frac{1}{2}\gamma M^2\delta p)^{\frac{\gamma-1}{\gamma}}$	$\bar{1}$·9604	$\bar{1}$·9655	$\bar{1}$·9702	$\bar{1}$·9742
$\log(1+\frac{\gamma-1}{2}M^2)$	0·0653	0·0653	0·0653	0·0653
$\log\left(\frac{c_1}{c_0}\right)^2$	$\bar{1}$·8951	$\bar{1}$·9002	$\bar{1}$·9049	$\bar{1}$·9089
$\left(\frac{c_1}{c_0}\right)^2$	0·7854	0·7947	0·8034	0·8108
$\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·3316	$\bar{1}$·3124	$\bar{1}$·2936	$\bar{1}$·2770
$(\gamma+1)^2\frac{c_1^2}{c_0^2}$	4·524	4·577	4·627	4·670
$\log\left[4\gamma-(\gamma+1)^2\frac{c_1^2}{c_0^2}\right]$	0·0319	0·0098	$\bar{1}$·9879	& $\bar{1}$·9685
$\gamma\log\frac{\gamma-1}{\gamma+1}$	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105	$\bar{2}$·9105
A	$\bar{2}$·9424	$\bar{2}$·9203	$\bar{2}$·8984	$\bar{2}$·8790
$\gamma\log(1-\frac{c_1^2}{c_0^2})$	$\bar{1}$·0642	$\bar{1}$·0374	$\bar{1}$·0110	$\bar{2}$·9878
$\log\frac{c_1^2}{c_0^2}$	$\bar{1}$·8951	$\bar{1}$·9002	$\bar{1}$·9049	$\bar{1}$·9089
$\log(\gamma^2-1)$	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823	$\bar{1}$·9823
B	$\bar{2}$·9416	$\bar{2}$·9199	$\bar{2}$·8962	$\bar{2}$·8790
A − B	0·0008	0·0004	0·0002	0·0000
C	1·4341	1·4341	1·4341	
$\log(A-B)$	$\bar{4}$·9031	$\bar{4}$·6021	$\bar{4}$·3010	
$\log(1-\frac{c_1^2}{c_0^2})^{1/2}$	$\bar{1}$·6658	$\bar{1}$·6562	$\bar{1}$·6468	
$\log(1+\frac{1}{2}\gamma M^2\delta p)^{1/\gamma}$	$\bar{1}$·9010	$\bar{1}$·9139	$\bar{1}$·9254	
	$\bar{3}$·9040	$\bar{3}$·6063	$\bar{3}$·3073	
Integrand. 10^3	8·017	4·039	2·029	

Hence, integrating, we find that

$$\delta C_D = 0{\cdot}0010 .$$

VI. On the oscillation of a cylinder in a viscous fluid contained in a fixed coaxal cylinder

Introduction

This problem, which arose in the course of an investigation into a certain method of measuring the viscosity of molten slags, may be stated as follows[1]. Viscous fluid is contained in a fixed cylinder and a second cylinder, coaxal with the first, is suspended in the fluid; find the couple (due to viscosity) exerted on the inner cylinder when it is constrained to execute a prescribed damped oscillation.

A search of the literature reveals that many problems of this type have been solved[2] (e.g. rotating disk, vibrating plate, cylinder rotating uniformly in an infinite fluid, etc.) but, as far as I am aware, no mention has yet been made of this particular problem.

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1. I am much indebted to Miss Helen Towers of the Department of Metallurgy, The Royal Technical College, Glasgow, for presenting me with this problem; the results obtained were incorporated by Miss Towers in her Ph.D. Thesis.
2. See Havelock: Phil Mag. (1921), 620
Meyer: Pogg. Ann. Bd 113 (1861), 555; Wied.Ann Bd. 32 (1887), 642.
Kobayashi: Zeits.f. Phys. 52 (1927), 448
Stokes: Collected Papers III, 1.

In the following lines the problem is solved by a perfectly straightforward and standard procedure.

We regard the cylinders as being of infinite length and, as is usually done in problems of this type, assume that the velocities are sufficiently small for their squares to be neglected.

Transformation of the Navier-Stokes Equations

Neglecting terms involving the squares of the velocities, the basic equations are

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \rho\frac{\partial u}{\partial t},$$

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho\frac{\partial v}{\partial t},$$

where u, v are the x, y components of the velocity V, p is the pressure, and μ, ρ are respectively the viscosity and density of the fluid. For the rotational motion contemplated

$$u = -V\sin\theta, \qquad v = V\cos\theta.$$

Since $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, we have

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial\theta}\sin\theta = \mu\left(-\sin\theta\frac{\partial^2 V}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial V}{\partial r} + \frac{V}{r^2}\sin\theta\right) + \rho\frac{\partial V}{\partial t}\sin\theta,$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial p}{\partial\theta}\cos\theta = \mu\left(\cos\theta\frac{\partial^2 V}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\cos\theta\right) - \rho\frac{\partial V}{\partial t}\cos\theta.$$

Now, from symmetry, $\frac{\partial p}{\partial\theta} = 0$; hence

$$\frac{\partial p}{\partial r} = -\tan\theta\left[\mu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) - \rho\frac{\partial V}{\partial t}\right]$$

$$= \cot\theta\left[\mu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) - \rho\frac{\partial V}{\partial t}\right].$$

Thus
$$\mu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) - \rho\frac{\partial V}{\partial t} = 0 \qquad (1)$$

and $\partial p/\partial r = 0$. Since $\partial p/\partial\theta = 0$, p is a constant throughout the field. (This is not surprising since the squares of the velocities have been neglected).

A more direct derivation of (1) is as follows. Consider an annular element of fluid of radius r and width δr, and let ω be the angular velocity of the fluid. Then the frictional force per unit area on a cylindrical shell of radius r is $2\pi r . r\mu\frac{\partial\omega}{\partial r}$, and the equation of motion of the annular element is
$$2\pi r^3 \rho\, \delta r\, \frac{\partial\omega}{\partial r} = \frac{\partial}{\partial r}\left(2\pi r^3 \mu \frac{\partial\omega}{\partial r}\right)\delta r$$
or
$$\mu\left(\frac{\partial^2\omega}{\partial r^2} + \frac{3}{r}\frac{\partial\omega}{\partial r}\right) = \rho\frac{\partial\omega}{\partial t}.$$
On putting $\omega = V/r$, we immediately obtain (1) from this equation.

Case of a cylinder rotating at constant speed in a fluid contained in a coaxal cylinder

We deal with this case first, for an application of Duhamel's Theorem to the result gives the result for the case in which the velocity of the inner cylinder is a function of the time.

If the radii of the inner and outer cylinders are a, b respectively, and if V_0 is the (constant) peripheral velocity of the inner cylinder, then we require a solution of (1), which satisfies the boundary conditions

(a) $V = 0$, for $r = b$ and all values of t

(b) $V = V_0$, when $r = a$ for all values of t

(c) $V = 0$, when $t = 0$ for all values of r.

Writing $\frac{\mu}{\rho} = \nu$, and putting $V = RT$ where R, T are pure functions of r, t respectively, we find that (1) gives

$$\nu T \left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} \right) = R \frac{dT}{dt},$$

whence $\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} \right) = \frac{1}{\nu T}\frac{dT}{dt} = \text{const.} = -\lambda^2.$

Thus $T = A'e^{-\nu\lambda^2 t}$, where A' is an arbitrary constant and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{1}{r^2}\right) R = 0.$$

The solution of this equation is

$R = B'J_1(\lambda r) + C'G_1(\lambda r)$, where B', C' are arbitrary, and where $J_1(x)$, $G_1(x)$ are Bessel Functions of the first order and respectively of the first and second kinds.

Hence, we have

$$V = e^{-\nu\lambda^2 t}\left\{A_\lambda J_1(\lambda r) + B_\lambda G_1(\lambda r)\right\}, \qquad (2)$$

where A_λ, B_λ are arbitrary constants, as a solution of (1). Any value of the constant λ gives such a solution; we shall, in this paper, be thinking of λ

as real. In the special case $\lambda = 0$ the solution turns out to be

$$V = Ar + \frac{B}{r} \quad , \qquad A, B \text{ arbitrary}$$

so that, a very general solution of (1) is

$$V = Ar + \frac{B}{r} + \sum_{\lambda} e^{-\nu\lambda^2 t}\{A_\lambda J_1(\lambda r) + B_\lambda G_1(\lambda r)\}, \qquad (3)$$

the values of λ, over which the summation is made, being not yet specified.

Consider the following special case of (3):

$$V = A\left(r - \frac{b^2}{r}\right) + \sum_{\lambda} A_\lambda e^{-\nu\lambda^2 t}\{J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a)\}, \qquad (4)$$

where the summation is over the positive roots of

$$J_1(\lambda a) G_1(\lambda b) - J_1(\lambda b) G_1(\lambda a) = 0. \qquad (5)$$

Then (4) satisfies boundary condition (2). It also satisfies (b) if

$$A = -aV_0/(b^2 - a^2).$$

We must now determine the A_λ so that

$$V = -\frac{aV_0}{b^2 - a^2}\left(r - \frac{b^2}{r}\right) + \sum_{\lambda} A_\lambda e^{-\nu\lambda^2 t}\{J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a)\}$$

satisfies condition (c), viz. when $t = 0$, $V = 0$ for all values of r, i.e. we must find the coefficients A_λ of the Fourier Bessel Expansion:

$$\frac{aV_0}{b^2 - a^2}\left(r - \frac{b^2}{r}\right) = \sum_{\lambda} A_\lambda \{J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a)\}, \qquad (6)$$

the summation being over the values of λ given by (5).

By a well-known theorem

$$A_\lambda = \frac{2\lambda^2 J_1^2(\lambda b)}{J_1^2(\lambda a) - J_1^2(\lambda b)} \int_a^b \frac{aV_0}{b^2-a^2}(r^2-b^2)\left\{ J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a) \right\} dr. \quad *$$

Since, for any Bessel Function $Z_n(x)$,

$$Z_0' = -Z_1, \quad \frac{d}{dx}(x^n Z_n) = x^n Z_{n-1},$$

we have

$$A_\lambda = \frac{aV_0}{b^2-a^2} \frac{2\lambda^2 J_1^2(\lambda b)}{J_1^2(\lambda a) - J_1^2(\lambda b)} \Big[\left\{ \frac{1}{\lambda} r^2 G_1(\lambda a) J_2(\lambda r) \right\}_a^b - \left\{ \frac{1}{\lambda} r^2 J_1(\lambda a) G_2(\lambda r) \right\}_a^b + \left\{ \frac{b^2}{\lambda} G_1(\lambda a) J_0(\lambda r) \right\}_a^b - \left\{ \frac{b^2}{\lambda} J_1(\lambda a) G_0(\lambda r) \right\}_a^b \Big].$$

Since $Z_2(x) + Z_0(x) = 2Z_1(x)/x$ and in virtue of (5), this expression for A_λ reduces to

$$A_\lambda = \frac{2aV_0 \lambda J_1^2(\lambda b)}{J_1^2(\lambda a) - J_1^2(\lambda b)} \left\{ J_1(\lambda a) G_0(\lambda a) - J_0(\lambda a) G_1(\lambda a) \right\}$$

$$= \frac{2aV_0 \lambda J_1^2(\lambda b)}{J_1^2(\lambda a) - J_1^2(\lambda b)} \left\{ -J_0'(\lambda a) G_0(\lambda a) + J_0(\lambda a) G_0'(\lambda a) \right\}.$$

Hence, since (MacRobert, p.284)

$$J_0'(\lambda a) G_0(\lambda a) - J_0(\lambda a) G_0'(\lambda a) = \frac{1}{\lambda a}, \qquad (7)$$

we have

$$A_\lambda = -\frac{2V_0 J_1^2(\lambda b)}{\left\{ J_1^2(\lambda a) - J_1^2(\lambda b) \right\}}. \qquad (8)$$

Thus,

$$V = V_0 \left[-\frac{a}{b^2-a^2}\left(r - \frac{b^2}{r}\right) - \sum_\lambda \frac{2J_1^2(\lambda b) e^{-\nu\lambda^2 t}}{J_1^2(\lambda a) - J_1^2(\lambda b)} \left\{ J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a) \right\} \right], \qquad (9)$$

or, if $\omega = \frac{V}{r}$, $\omega_0 = \frac{V_0}{a}$,

$$\omega = \omega_0 \left[\frac{a^2}{b^2-a^2}\left(\frac{b^2}{r^2} - 1\right) - \sum_\lambda \frac{2a J_1^2(\lambda b) e^{-\nu\lambda^2 t}}{r\left\{ J_1^2(\lambda a) - J_1^2(\lambda b) \right\}} \left\{ J_1(\lambda r) G_1(\lambda a) - G_1(\lambda r) J_1(\lambda a) \right\} \right]. \qquad (10)$$

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* MacRobert: Spherical Harmonics, 1st ed. p. 284.

<u>Case of inner cylinder rotating with variable velocity</u>

If now, ω_0 is not constant, but is a function of the time $\omega_0(t)$, Duhamel's Theorem, applied to (10) yields

$$\omega = \int_0^t \omega_0'(\tau)\left[\frac{a^2}{b^2-a^2}\left(\frac{b^2}{r^2}-1\right) - \sum_\lambda \frac{2a\, J_1^2(\lambda b)e^{-\nu\lambda^2(t-\tau)}}{r\{J_1^2(\lambda a) - J_1^2(\lambda b)\}} \times \{J_1(\lambda r)G_1(\lambda a) - G_1(\lambda r)J_1(\lambda a)\}\right]d\tau. \quad (11)$$

We now give an alternative and much simpler derivation of (11) using Hankel Transforms as introduced by Sneddon [1.]

<u>Alternative Derivation of (11)</u>

If $v(r)$ is a function of r, defined over the range $a < r < b$, then its finite Hankel Transform is given by

$$\overline{v}_H = \int_a^b r\, v(r)\{J_1(\lambda r)G_1(\lambda a) - J_1(\lambda a)G_1(\lambda r)\}dr,$$

where λ is any of the quantities defined by (5).

It can then be shown that

$$v(r) = \sum_\lambda \frac{2\lambda^2 J_1^2(\lambda b)\,\overline{v}_H}{J_1^2(\lambda a) - J_1^2(\lambda b)}\{J_1(\lambda r)G_1(\lambda a) - G_1(\lambda r)J_1(\lambda a)\}.$$

(cf. the above work, or MacRobert, loc. cit., p.284)

Further, using various recurrence formulae for Bessel Functions, it can be shown that

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1. Sneddon: Phil. Mag. ser. 7, 37, (1946),17. I had already derived (11) by the above standard method before this paper was published.

$$\int_a^b r\left\{J_1(\lambda r)G_1(\lambda a)-J_1(\lambda a)G_1(\lambda r)\right\}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v(r)\,dr$$

$$=-\lambda^2\bar{V}_H+v(a)-v(b)\frac{J_1(\lambda a)}{J_1(\lambda b)}.$$

(This formulae is the correct form of the one given by Sneddon; in his paper the coefficient of $v(a)$ is erroneous).

From (1) it follows that

$$\int_a^b r\left\{J_1(\lambda r)G_1(\lambda a)-J_1(\lambda a)G_1(\lambda r)\right\}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v(r)\,dr$$
$$=\frac{1}{\nu}\frac{\partial}{\partial t}\int_a^b r\left\{J_1(\lambda r)G_1(\lambda a)-J_1(\lambda a)G_1(\lambda r)\right\}v(r)\,dr.$$

Now, $v(a)=a\omega_0(t)$, $v(b)=0$, whence

$$-\lambda^2\bar{V}_H+a\omega_0(t)=\frac{1}{\nu}\frac{\partial\bar{V}_H}{\partial t}.$$

Hence,

$$\bar{V}_H e^{\nu\lambda^2 t}=\int_0^t e^{\nu\lambda^2\tau}a\nu\,\omega_0(\tau)\,d\tau$$

or

$$\bar{V}_H=\int_0^t e^{-\nu\lambda^2(t-\tau)}a\nu\,\omega_0(\tau)\,d\tau.$$

Thus,

$$v=\sum_\lambda\frac{2\lambda^2 a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\}\int_0^t e^{-\nu\lambda^2(t-\tau)}\nu\,\omega_0(\tau)\,d\tau$$

$$=\sum_\lambda\frac{2\lambda^2 a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\}\left[\left\{\frac{1}{\lambda^2}e^{-\nu\lambda^2(t-\tau)}\omega_0(\tau)\right\}_0^t-\frac{1}{\lambda^2}\int_0^t e^{-\nu\lambda^2(t-\tau)}\omega_0'(\tau)\,d\tau\right]$$

$$=\sum_\lambda\left[\frac{2a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\}\omega_0(t)\right.$$
$$\left.-\int_0^t\frac{2a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\}e^{-\nu\lambda^2(t-\tau)}\omega_0'(\tau)\,d\tau\right].$$

It can easily be shown (cf. the definition and inversion of the finite Hankel Transform and (6) (8) above) that

$$\frac{a^2}{b^2-a^2}\left(\frac{b^2}{r}-r\right) = \sum_{\lambda} \frac{2a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\},$$

so that

$$v(r) = \frac{a^2\omega_0(t)}{b^2-a^2}\left(\frac{b^2}{r}-r\right)-\sum_{\lambda}\int_0^t \frac{2a\,J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\} \times e^{-\nu\lambda^2(t-\tau)}\,\omega_0'(\tau)\,d\tau.$$

This equation is equivalent to (11).

<u>The couple acting on the inner cylinder</u>

Integrating by parts, we immediately have from (11)

$$\omega = \frac{a^2}{b^2-a^2}\left(\frac{b^2}{r^2}-1\right)\omega_0(t) - \sum_{\lambda}\frac{2a\,J_1^2(\lambda b)}{r\{J_1^2(\lambda a)-J_1^2(\lambda b)\}}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\} \times\left[\omega_0(t)-\nu\lambda^2\int_0^t \omega_0(\tau)e^{-\nu\lambda^2(t-\tau)}\,d\tau\right]$$

since $\omega_0(0) = 0$.

Hence, by (6) and (8)

$$\omega = \sum_{\lambda}\frac{2a\nu\lambda^2 J_1^2(\lambda b)}{r\{J_1^2(\lambda a)-J_1^2(\lambda b)\}}\left\{J_1(\lambda r)G_1(\lambda a)-G_1(\lambda r)J_1(\lambda a)\right\} \times\int_0^t \omega_0(\tau)e^{-\nu\lambda^2(t-\tau)}\,d\tau, \qquad (12)$$

so that

$$\frac{d\omega}{dr} = \sum_{\lambda}\frac{2a\nu\lambda^3 J_1^2(\lambda b)}{r\{J_1^2(\lambda a)-J_1^2(\lambda b)\}}\left\{J_1'(\lambda r)G_1(\lambda a)-G_1'(\lambda r)J_1(\lambda a)\right\}\int_0^t \omega_0(\tau)e^{-\nu\lambda^2(t-\tau)}\,d\tau$$

+ term which vanishes when $r = a$.

Thus

$$\left(\frac{d\omega}{dr}\right)_{r=a} = \sum_{\lambda}\frac{2\nu\lambda^3 J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\,\frac{1}{\lambda a}\int_0^t \omega_0(\tau)e^{-\nu\lambda^2(t-\tau)}\,d\tau,$$

i.e.

$$\left(\frac{d\omega}{dr}\right)_{r=a} = \frac{2\nu}{a}\sum_{\lambda}\frac{\lambda^2 J_1^2(\lambda b)}{J_1^2(\lambda a)-J_1^2(\lambda b)}\int_0^t \omega_0(\tau)e^{-\nu\lambda^2(t-\tau)}\,d\tau. \qquad (13)$$

Now the couple M on the inner cylinder is given by

$$M = 2\pi\mu a^3 \left(\frac{d\omega}{dr}\right)_{r=a},$$

so that

$$M = 4\pi\mu\nu a^2 \sum_\lambda \frac{\lambda^2 J_1^2(\lambda b)}{J_1^2(\lambda a) - J_1^2(\lambda b)} \int_0^t \omega_0(\tau) e^{-\nu\lambda^2(t-\tau)} d\tau. \tag{14}$$

Let us suppose now that the angular velocity of the inner cylinder is prescribed to be

$$\omega_0(\tau) = \Omega e^{-\kappa\tau} \sin p\tau.$$

Then
$$\int_0^t \omega_0(\tau) e^{-\nu\lambda^2(t-\tau)} d\tau = \Omega e^{-\nu\lambda^2 t} \int_0^t e^{-(\kappa-\nu\lambda^2)\tau} \sin p\tau \, d\tau$$

$$= -\frac{\Omega e^{-\kappa t}}{(\kappa-\nu\lambda^2)^2 + p^2}\left\{(\kappa-\nu\lambda^2)\sin pt + p\cos pt\right\} + \frac{\Omega p e^{-\nu\lambda^2 t}}{(\kappa-\nu\lambda^2)^2 + p^2}, \tag{15}$$

and

$$M = 2\pi\Omega\mu a^3 \sum_\lambda \left[P_\lambda e^{-\nu\lambda^2 t} - e^{-\kappa t}\left(Q_\lambda \sin pt + R_\lambda \cos pt\right)\right], \tag{16}$$

where for brevity, P_λ, Q_λ, R_λ are used to denote functions of λ, a, b, ν, p, κ which can be easily obtained from (14) and (15). (16) provides the solution of the problem.

A problem related to the one considered above, but much more difficult, is as follows. Let the inner cylinder be suspended under torsion, then released and allowed to oscillate in the fluid until it comes to rest; find the relationship between the damping factor of the oscillations, the viscosity of the fluid and the mechanical constants of the system.

In this case, there are no external forces acting on the cylinder except those due to the torsion of the wire suspending it and to the viscosity of the fluid, and so the equation of motion is

$$I\ddot{\theta} + M + G\theta = 0, \tag{17}$$

where θ is the angular displacement of the cylinder from its equilibrium position, I is the moment of inertia of the cylinder about its axis and G depends on the torsional system. $\omega_0(t)$ is now the unknown quantity and ~~is~~ given by the integro-differential equation obtained by substituting in (17) the expression for M given by (14). I have tried to solve this equation by means of the operational calculus and also by transforming it to an integral equation and then using Whittaker's method, but without success.

ADDITIONAL PAPERS.

THE FLUXGATE PRINCIPLE.

1. Introduction. In 1936 Aschenbrenner and Goubau [1] devised a very sensitive magnetometer in which a novel method was employed to measure the strength of an external magnetic field, and, since that time, there have appeared several instruments (mostly aircraft compasses) which work on essentially the same principle. Each system includes one or more coils with ferromagnetic cores, round which alternating currents are flowing, and the inductive effect of the external magnetic field on these cores causes changes in the alternating currents (and in their associated E.M.F.'s) in certain elements of the system; under certain conditions these changes are directly proportional to the strength of the field. For instance, the magnetic field may induce in one of the secondary currents or voltages a second harmonic, whose amplitude is proportional to the field strength. Instruments operating in this way have come to be described as working on the Fluxgate Principle+ although, of course, there is no new principle involved, in the strict sense of the term.

+ The term "fluxgate" appears in the trade name of various products of the Pioneer Instrument Division of the Bendix Aviation Corporation. As will be seen later, the designation is singularly appropriate.

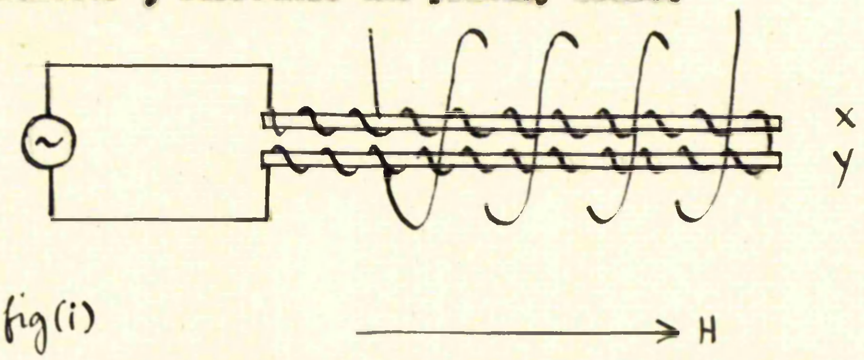
As far as I am aware, no adequate explanation of the principle has yet been given, although a somewhat limited account was given by the two German authors, who were well aware of its shortcomings. In some of the literature, the linear relationship between magnetic field strength and change of current or voltage is simply stated as a fact. However, it will be seen below that the relationship is not perfectly linear, and that it is only by a suitable choice of certain parameters of the system that the approximation to linearity can be made sufficiently close to be of practical value. The object of this paper is to put the principle on a rather more secure theoretical foundation by giving a fairly detailed treatment of the fluxgate type of magnetometer, which is the simplest system, and forms the basis on which the other more complicated systems are built.

The paper concludes with an account of some of the more important applications of the Fluxgate Principle.

2. Physical Explanation of the operation of the magnetometer.

The apparatus (see fig. (i)) consists essentially of two identical rods X, Y of high permeability ferromagnetic material (e.g., mumetal or permalloy), wound in such a way that they are magnetized in opposite directions when a current flows. This current is maintained by a source of alternating E.M.F.; a secondary/

secondary/ coil in series with a large resistance and a voltmeter[/], surrounds the primary coils.



fig(i)

The operation of the instrument depends on the fact that when the system is placed in a magnetic field whose strength in the direction of the rods is H, the secondary voltage is of twice the frequency of the applied voltage, and has an amplitude which is proportional to H to a close approximation.

In practice, the form of the applied voltage is maintained (e.g., purely sinusoidal) and that of the exciting current allowed to vary, but in theory it is much simpler to regard the exciting current as being maintained in form, and the form of the applied voltage as being allowed to vary. Nothing of importance physically is lost by doing this, but a considerable mathematical simplification is gained. We assume then that the exciting current is purely sinusoidal.

[/] In practice, the secondary voltage is small, and is amplified before being measured.

The manner in which the second harmonic arises in the secondary voltage can be seen from the following considerations.* The resultant magnetizing force on the bar X is $H + H_1$, where H_1 is the strength of the magnetic field due to the exciting current. By means of the B - H curve for the material of the core, the wave of flux density in the bar can be derived (see fig. (ii)), and this, by a change of scale can be made to represent the variation of flux interlinkages with the secondary coil due to the bar X. Similarly, the curves for bar Y can be obtained and these are represented in the figure by dotted lines. The curve obtained by adding the two waves gives the total number of interlinkages through the secondary coil, and the induced voltage is obtained by plotting the negative of the gradient of this curve; it is evident that this voltage has twice the frequency of the exciting current. It is also clear why the designation "fluxgate" is appropriate. When the cores are saturated, the external magnetic field has practically no effect on the flux density, so that twice in each cycle of the exciting current, the external field is excluded; the degree of saturation of the cores acts as a gate for letting it in or shutting it out. It is obvious that, for the satisfactory operation of the instrument, the exciting current must have sufficient amplitude to saturate the cores twice in each cycle.

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* The argument of this section is due to Mr. I. L. Thomas, B.Sc., A. M. I. E.E.

It is not obvious that the amplitude of the secondary voltage is very nearly proportional to the magnetic field strength; this will be established in § 4. The next section is devoted to a few preliminaries required for the subsequent theory.

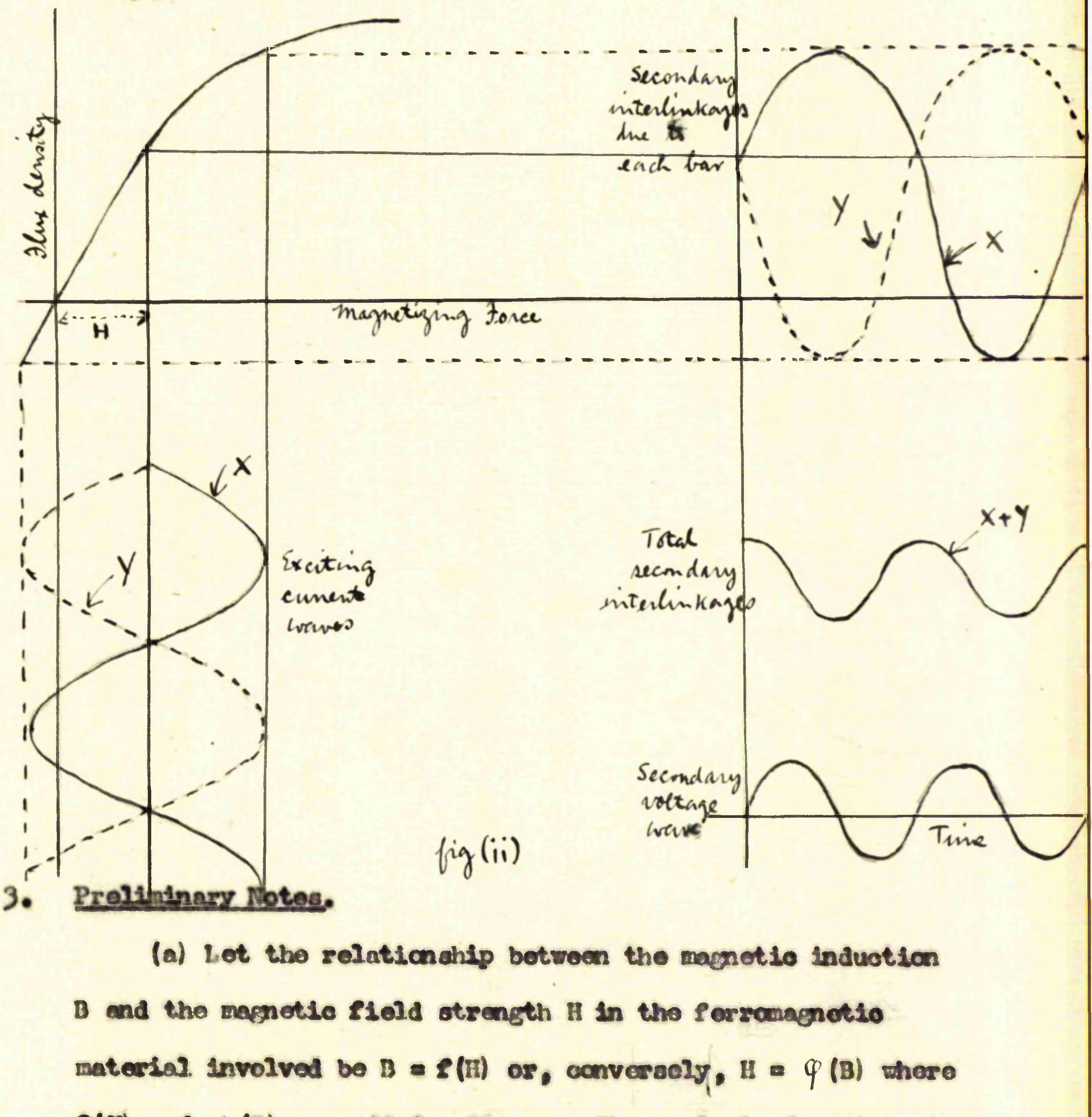

fig (ii)

3. Preliminary Notes.

(a) Let the relationship between the magnetic induction B and the magnetic field strength H in the ferromagnetic material involved be $B = f(H)$ or, conversely, $H = \varphi(B)$ where $f(H)$ and $\varphi(B)$ are odd functions. The neglect of hysteresis is almost unavoidable but, as the hysteresis loop is small for

for/ mumetal or permalloy, little error can accrue by taking a mean B - H relationship.

Since f(H) and φ(B) are odd functions, their odd derivatives are even functions and their even derivatives are odd functions. Also, since

$$f\{\varphi(x)\} = x,$$

then $$f'\{\varphi(x)\} = \frac{1}{\varphi'(x)}. \qquad (3.1)$$

(b) When a ferromagnetic rod of finite rather than infinite length is considered, the self-demagnetizing effect, due to the free magnetism at the ends, must be taken into account. If the strength of the external field is H, then the strength of the field actually operative in the rod is given by

$$H' = H - NI$$

where I is the intensity of magnetization and N is a constant. Besides depending on the shape of the rod and on the direction of magnetization, N is influenced by the pattern of the lines of magnetizing force outside the bar. For instance, in the case of the magnetometer (fig.(i)), the lines of force due to the exciting current tend to form closed circuits through the two rods, while those due to the external field have no such tendency. Thus, the value of N for the magnetization due to the exciting current is much smaller than that for the magnetization due to the external field; if there were no external field, the two straight rods, very close together, would behave almost like a closed ring for which N = 0.

In this paper, we are especially interested in the case in which N is sufficiently large for H' to correspond to the initial straight part of the B - H curve (for values of H up to 0·4 oersted) and we wish to express H' in terms of H. Since

$$B = f(H') = H' + 4\pi I$$

and $$f(H') \doteqdot \mu_0 H',$$

where μ_0 is the initial permeability, we have by (3.2)

$$I \doteqdot \frac{(\mu_0 - 1)H}{4\pi + (\mu_0 - 1)N} \qquad (3.3)$$

and $$H' \doteqdot \frac{4\pi H}{4\pi + (\mu_0 - 1)N}. \qquad (3.4)$$

Since μ_0 is very large compared with unity, $\mu_0 - 1$ may be replaced by μ_0 in these formulae.

(c) The assumption is made throughout that ohmic resistance is negligible. In some of the instruments, which we shall have in mind, the reactance is about ten times ohmic resistance, so that neglect of the latter alters the impedance by only $\frac{1}{2}\%$.

4. Mathematical Theory of the Magnetometer.

Let A be the cross sectional area of each rod and let the number of turns per unit length of each primary coil and of the secondary coil be n and m respectively.

As mentioned in § 2, it turns out to be simpler to regard the exciting current, rather than the applied E.M.F. as having a fixed sinusoidal form; accordingly, let the exciting current be $i = i_0 \sin \omega t$. Let H_1, H denote the strengths of the magnetic field due to the current and the external field respectively, and let H_i' and H' be the corresponding field strengths actually

actually/ operative in the rods. Then the flux density in bar X is $f(H'+H_i')$ and in bar Y is $f(H'-H_i')$.

Consequently, by Lenz's Law, the induced E.M.F. in the secondary coil is given by

$$V = mA\frac{d}{dt}\left\{f(H_i'+H') + f(-H_i'+H')\right\}$$

$$= mA\frac{d}{dt}\left\{f(H_i'+H') - f(H_i'-H')\right\}, \qquad (4.1)$$

since f(x) is an odd function of x.

When H = H' = 0, the induced voltage in the secondary coil is, of course, zero.

Expanding the right hand side of (4.1) by Taylor's Theorem, we obtain

$$V = 2mA\frac{d}{dt}\left\{H'f'(H_i') + \frac{H'^3}{3!}f'''(H_i') + \cdots\right\}$$

$$= 2mAH'\frac{dH_i'}{dt}\left\{f''(H_i') + \frac{H'^2}{3!}f^{(iv)}(H_i') + \cdots\right\}. \qquad (4.2)$$

Suppose now that H' is sufficiently small for the second term in the expansion to be negligible compared with the first; the condition for this to be so will be discussed later. Then by (3.4)

$$V = \frac{8\pi mAH}{4\pi+(\mu_0-1)N}\frac{dH_i'}{dt}f''(H_i'), \qquad (4.3)$$

i.e., the E.M.F. induced in the secondary coil is proportional to the component of the external magnetic field in the direction of the rods. The value of N to be inserted in (4.3) is the one corresponding to the lines of force of the external field. The value of N which relates H_i' to H_i is appreciably/

appreciably/ smaller, and, if the two rods are very close together, will be nearly zero. In that case $H_i' \doteq H_i = 4\pi n i_0 \sin\omega t/10$, whence, since f(x) is an odd function of x, the leading term on the right hand side of (4.3) is a second harmonic. The exact evaluation of H' cannot be easily carried out without a knowledge of the precise form of f(x).

We consider now the rejected term in the expansion (4.2), and for this purpose require an explicit form for f(x). The following formula has been derived* by the Method of Least Squares from experimentally determined values of B and H in the mumetal toroid;

$$B = \frac{H}{\sqrt{(a + bH^2)}}, \quad \text{where} \quad \begin{array}{l} a = 1{\cdot}22 \,.\, 10^{-9}, \\ b = 1{\cdot}49 \,.\, 10^{-8}. \end{array} \qquad (4.4)$$

On the basis of this formula, we estimate the relative magnitudes of the terms rejected and retained in (4.2) by evaluating $f^{(iv)}(H_i')/f''(H_i')$; the ratio of the amplitudes turns out (after considerable labour) to be approximately $12b/a$, i.e. ~ 150. If we now decide that the approximation (4.3) must be correct to within 1%, then it is necessary that

$$\frac{150}{3!} H'^2 \leqq 0{\cdot}01$$

$$\text{ie} \quad H' \leqq 0{\cdot}02.$$

This value corresponds to the initial straight part of the B - H curve, so that the use of (3.4) is justified.

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*See Appendix I.

Further, suppose that the instrument is required to measure values of H up to 0.4 (the maximum value of the horizontal component of the earth's magnetic field), then N must be sufficiently large to reduce the field inside the cores from 0.4 to 0.02. Hence, by (3.4)

$$\frac{4\pi .\ 0\cdot 4}{4\pi + 28,000\ N} \leqq 0\cdot 02$$

$$\text{or} \quad \frac{N}{4\pi} \geqq 0\cdot 0007.$$

Tables of $N/4\pi$ for cylindrical rods of given length/diameter ratio are to be found in the International Critical Tables. Hence we find that the above condition is equivalent to demanding that the length/diameter ratio of the mumetal rods must not exceed 70:1. This value is only approximate, and, in any case, its precise value depends on the magnitude of the field strengths contemplated, and on the accuracy of the measurement required. Thus the approximation (4.3) is justified provided that the length to diameter ratio of the rods does not exceed a certain critical value; broadly speaking this value may be taken as 100:1 in the case of the magnetometry of the earth's field.

We now consider the system (like the actual magnetometer) in which the applied voltage is given as $E \sin \omega t$ and the exciting current i is left unspecified; we shall not however make any numerical considerations for, in this case, these are very complicated. With the same notation as before,

before,/ Lenz's Law applied to the primary and secondary circuits gives

$$nA\frac{d}{dt}\left\{f(H_i'+H')+f(H_i'-H')\right\} = -E\sin\omega t \qquad (4.5)$$

$$V = mA\frac{d}{dt}\left\{f(H_i'+H')-f(H_i'-H')\right\}. \qquad (4.6)$$

We wish to express V in terms of H and the constants of the system, and shall do this by means of a perturbation method. The solution for the special case of zero external field can be obtained exactly, whence, on the assumption that the presence of the external field modifies the currents and voltages only slightly, a close approximation to the general solution can be obtained.

When $H = H' = 0$, let H_i' have the value H_{io}' ; then (4.6) is nugatory and (4.5) yields

$$2An\frac{d}{dt}\left\{f(H_{io}')\right\} = -E\sin\omega t,$$

whence $$f(H_{io}') = \frac{E}{2\omega An}\cos\omega t + C, \qquad (4.7)$$

where C is the constant of integration. Now, since the average value of the applied E.M.F. is zero, the same must be true of the current i for, once the steady state has been reached, there is no E.M.F. available to drive even a small direct current against even an infinitesimal resistance. Hence, the average values of H_{io} and H_{io}' are also zero. Therefore since f is an odd function, the average value of the left hand side of (4.7) is zero and so C must be zero. Thus

$$H_{io}' = \varphi(E\cos\omega t/2\omega An). \qquad (4.8)$$

When the external field is non-zero, let $H_i' = H_{i0}' + \Sigma$; then (4.6) gives

$$V = mA \frac{d}{dt}\left\{ f(H_{i0}' + \Sigma + H') - f(H_{i0}' + \Sigma - H') \right\}. \qquad (4.9)$$

Assuming that $\Sigma \pm H'$ is small, we expand the right hand side of this equation by Taylor's Theorem and, retaining only the first two terms of each expansion, obtain

$$V = 2mAH' \frac{d}{dt}\left\{ f'(H_{i0}') \right\}.$$

Hence, by (3.1)

$$V = \frac{m}{n} H'E \frac{\varphi''\left(\frac{E\cos\omega t}{2\omega An}\right)}{\left\{\varphi'\left(\frac{E\cos\omega t}{2\omega An}\right)\right\}^2} \sin\omega t.$$

Finally, expressing H' in terms of H by (3.4), we arrive at the result

$$V = \frac{HEm}{n(4\pi + \mu_0 N)} \frac{\varphi''(E\cos\omega t/2\omega An)}{\left\{\varphi'(E\cos\omega t/2\omega An)\right\}^2} \sin\omega t. \qquad (4.10)$$

Since φ is an odd function, it follows that V has period $\frac{\pi}{\omega}$ and so V consists solely of <u>even</u> harmonics, whose amplitudes are proportional to H. The length/diameter ratio of the rods will be limited as before if a close approximation to linearity is required.

(4.10) has the advantage over the corresponding formula (4.3) in that the constant of proportionality is explicitly given.

5. Some further applications of the fluxgate principle.

(1) The Pioneer Gyro Fluxgate Compass.[2]

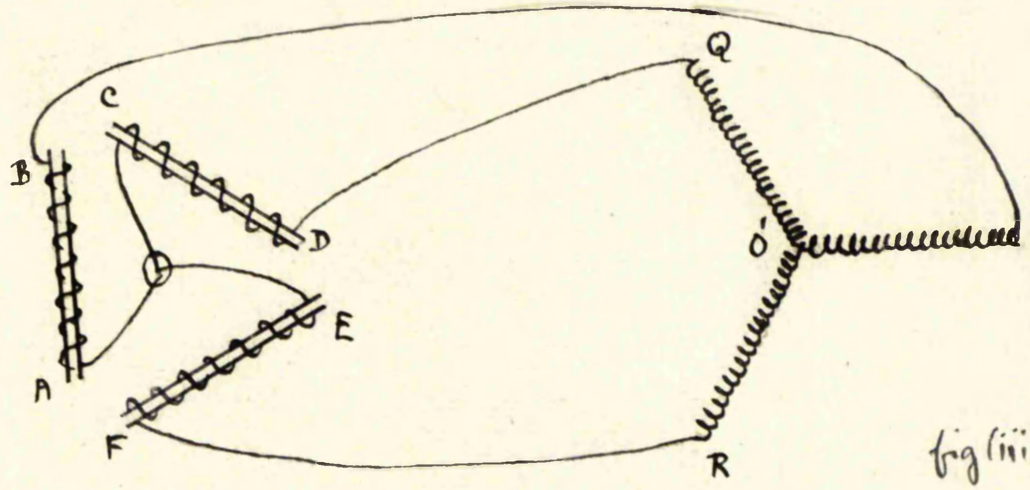


fig (iii)

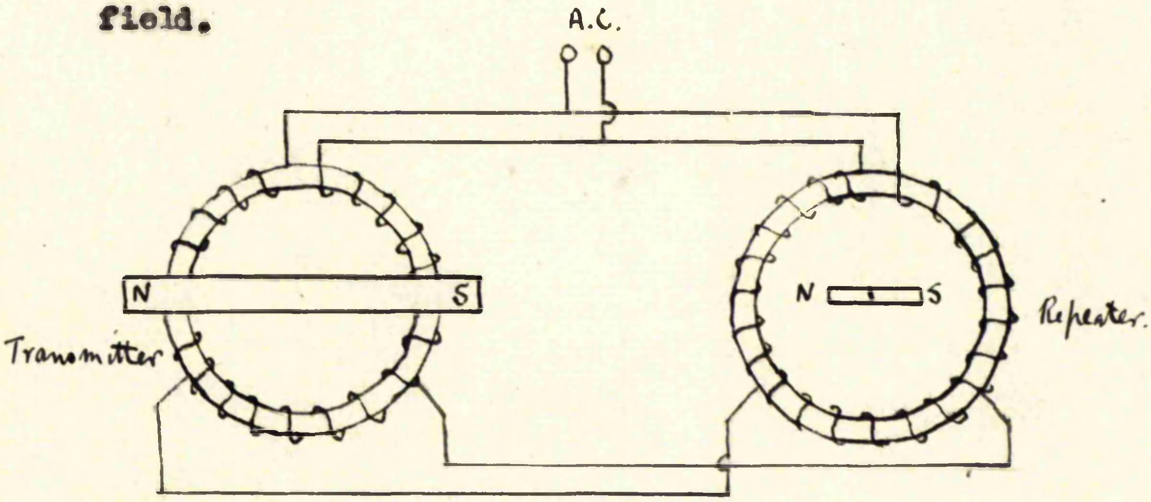
(the exciting windings are omitted from the diagram)

In this remote-indicating compass three pairs of fluxgate elements AB, CD and EF are arranged in the form of an equilateral triangle (fig.(iii)) and are exposed to the earth's field. Secondary coils surround each pair of elements and these are led, as shown in the figure, to three coils O'P, O'Q and O'R with ferromagnetic cores placed 120° appart; the system O'(PQR) is shielded from the earth's field. If the components of the earth's Magnetic field along AB, CD and EF are H_1, H_2 and H_3 respectively, then it can be shown that the secondary currents, and therefore the currents in O'P, O'Q and O'R, are proportional to H_1, H_2 and H_3, respectively. Hence, if the system O'(PQR) is suitably oriented, a magnetic field parallel to that of the earth is generated at O'. The complete theory of the instrument can be worked out in exactly the same way as for the magnetometer.

This instrument is designed for aircraft use, in which case the transmitter unit is placed in a part of the aircraft free from extraneous magnetic fields and the shielded repeater system is placed in the pilot's cockpit

(2) The Magnesyn Transmitter - Repeater System.

In the case of this instrument, the fluxgate principle is used in an unusual manner, viz., after a unidirectional magnetic field has been used to introduce even harmonics into a system, these harmonics are then used elsewhere to produce a unidirectional field.



fig(iv)

The system consists of two coils with mumetal cores (which we shall assume to be identical) and with magnets pivoted at their centres. A source of alternating E.M.F. is included and interconnecting leads are taken from the coils at 120° intervals as shown in the diagram.

The application of this system to Remote Indicating Compasses for aircraft rests on the fact that when the magnets are in equilibrium they are aligned parallel to each other. In an actual compass, one coil (the transmitter) is placed in the/

the/ tail or wing of the aircraft and far from any extraneous magnetic fields, and is exposed to the earth's field, while the other coil (the repeater) is near the pilot's dashboard and is magnetically shielded.

An attempt to give a rough quantitative explanation of the operation of the system is made in Appendix II.

References.

(1) Aschenbrenner & Goubau : Hochfrequentztechnik u. Electroakustic Vol. 47, No.6 (1936), 177 - 181.

(2) See: Automotive & Aviation Industries, 89, No. 9 (1944).

(3) Smith: Aeronautical Engineering Review, May 1934, 31 - 36.

Appendix I - The B - H Curve for Mumetal.

We seek to determine a relationship of the form

$$B = \frac{H}{\sqrt{(a + bH^2)}}$$

from the following values of B and H measured* in a mumetal toroid:

B	4400	5450	6100	6500	6750	7250	7500	7800	7910	8300
H	0·1	0·2	0·3	0·4	0·5	0·75	1·0	1·5	2·0	4·0

The corresponding values of H^2 and $\frac{H^2}{B^2}$ are then as follows

H^2	·01	0·04	0·09	0·16	0·25	0·56	1·0	2·25	4·0	16·0
$10^{10}\frac{H^2}{B^2}$	5·16	13·47	24·18	37·87	54·88	107·0	177·7	369·8	639·2	2323

We then assume that

$$10^{10}\frac{H^2}{B^2} = a + bH^2$$

and determine the best values of a and b by the method of Least Squares.

The normal equations turn out to be

$$10a + 24{\cdot}36b = 3750\ ,$$

$$24{\cdot}36a + 280b = 42{,}000,$$

whence $a = 12{\cdot}18, \quad b = 149$

Thus $$B = \frac{10^5 H}{\sqrt{(12{\cdot}18 + 149 H^2)}}.$$

The values of B corresponding to the values of H in the above table can then be calculated by this formula and compared with the observed values. It can be seen from the corresponding/

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* The measurements were made by Mr. I. L. Thomas at the Royal Aircraft Establishment, Farnborough.

corresponding graphs that the formula fits the data reasonably well.

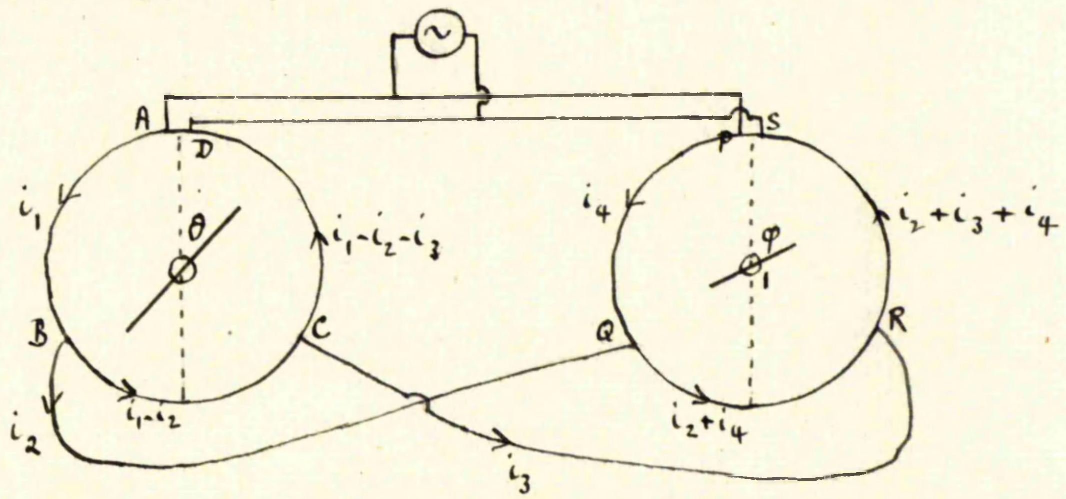
A formula of the form

$$B = \frac{H}{a + bH} \quad , \qquad H > 0 \, ,$$

fits the data better, but it is of no use to us here since it does not represent B as an odd function of H.

Appendix II. The Magnesyn Transmitter - Repeater System.

An attempt is made here to give a rough quantitative explanation of the operation of the Magnesyn System. Some rather crude approximations are made in the course of the work, for a perfectly rigorous treatment of the problem would be exceedingly complicated. For simplicity, too, certain constants of proportionality (e.g. number of turns in the coils etc.) have been omitted; they could be easily introduced at the end if desired. The theory is therefore far from complete and no pretence at finality is made; the work is a step in the right direction but nothing more. However, as far as I am aware, no attempt has ever been made at a theoretical treatment of the problem.



We consider the idealized case in which the alternator maintains an E.M.F. of $E \sin \omega t$, and before considering the effect of the earth's field, we prove that the two magnets when left to themselves must be parallel to each other.

The currents, four of which are independent, are designated as in the diagram. When there are no magnets present, the/

the same current flows through each sector of each coil and by Lenz's Law, is given by

$$-\frac{d}{dt}f(i) = \tfrac{1}{3}E\sin\omega t.$$

The number of turns in any one sector of a ring is taken as unity, so that in this case when the current flows along all three sectors the number of turns must be taken as 3. Hence

$$f(i) = \frac{E}{3\omega}\cos\omega t + C,$$

where C is the constant of integration; as in (4.8) above C vanishes. Thus

$$i = \varphi\left(\frac{E}{3\omega}\cos\omega t\right). \tag{1}$$

Suppose now that the magnets are pivoted at the centre of the coils, and that their axes make angles θ and φ with OA and OP respectively. Further, imagine that in the sectors AB, BC, CD, PQ, QR, RS of the coils there are respectively $m_1, m_2, m_3, M_1, M_2, M_3$ lines of force due to the magnets. To be more precise, m_1 say is the average value of the tangential component of the magnetic field strength along AB. For instance, if the coils are of radius a, and if the magnets are dipoles (doublets) of moments μ, μ' then

$$m_1 = \frac{3\mu}{4\pi a^3}\left(-3\cos\theta - \sqrt{3}\sin\theta\right),$$

$$m_2 = \frac{3\mu}{4\pi a^3}\cdot 2\sqrt{3}\sin\theta, \tag{2}$$

$$m_3 = \frac{3\mu}{4\pi a^3}\left(3\cos\theta - \sqrt{3}\sin\theta\right),$$

with similar expressions for M_1, M_2, M_3 involving μ', φ in place of μ, θ. These expressions are derived from the well-known expression $\mu\cos\theta/r^2$ for the potential of a dipole

dipole/ at a point distant r from it in a direction making an angle θ with its axis.

An immediate consequence of (2) is that $m_1 + m_2 + m_3 = 0$ as of course it ought to be, if this method of approximation is to be self-consistent.

Since A, B, C, D are at the same potentials as P, Q, R, S respectively, then

$$\frac{d}{dt}\{f(i_1 + m_1)\} = \frac{d}{dt}\{f(i_4 + M_1)\},$$

$$\frac{d}{dt}\{f(i_1 - i_2 + m_2)\} = \frac{d}{dt}\{f(i_2 + i_4 + M_2)\}, \qquad (3)$$

$$\frac{d}{dt}\{f(i_1 - i_2 - i_3 + m_3)\} = \frac{d}{dt}\{f(i_2 + i_3 + i_4 + M_3)\},$$

and sum of the terms on either side $= - E \sin \omega t$.

Now, when there are no magnets present,

$$i_1 = i_4 = \varphi\left(\frac{E}{3\omega}\cos\omega t\right),$$

$$i_2 = i_3 = 0,$$

by (1).

However, after the magnets are inserted, let

$$i_1 = i + \varepsilon_1, \quad i_2 = \varepsilon_2, \quad i_3 = \varepsilon_3, \quad i_4 = i + \varepsilon_4, \qquad (4)$$

it being supposed that the ε's are small. This is known to be true provided the magnets are not too far out of alignment with each other, and provided there is not too great a disparity between the powers of the magnets.

The first of equations (3) in conjunction with (4), gives

$$\frac{d}{dt}\{f(i + m_1 + \varepsilon_1) - f(i + M_1 + \varepsilon_4)\} = 0$$

ie $$\frac{d}{dt}\{(m_1 + \varepsilon_1 - M_1 - \varepsilon_4) f'(i)\} = 0$$

on expanding by Taylor's Theorem, and rejecting all terms beyond/

beyond/ the second. Integrating

$$\frac{m_1 + \varepsilon_1 - M_1 - \varepsilon_4}{\varphi'\left(\frac{E}{3\omega}\cos\omega t\right)} = C$$

by (3.1); C is a constant of integration.

Hence,

$$\varepsilon_1 - \varepsilon_4 + m_1 - M_1 = C\left\{\tfrac{1}{2}a_0 + a_2\cos 2\omega t + a_4\cos 4\omega t + \cdots\right\} \quad (5)$$

where $\frac{1}{2}a_0, a_2, a_4, \ldots$ are the Fourier Co-efficients of $\varphi'\left(\frac{E}{3\omega}\cos\omega t\right)$ and which can be calculated for any given function f.

To ensure that $\varepsilon_1, \varepsilon_4$ contain no constant part, C must be chosen such that

$$m_1 - M_1 = \tfrac{1}{2}Ca_0$$

$$\text{ie} \qquad C = \frac{2}{a_0}(m_1 - M_1) \qquad (6)$$

Thus $$\varepsilon_1 - \varepsilon_4 = \frac{2}{a_0}(m_1 - M_1)(a_2\cos 2\omega t + a_4\cos 4\omega t + \cdots).$$

Similarly, the remaining equations in (3) give

$$\varepsilon_1 - 2\varepsilon_2 - \varepsilon_4 = \frac{2}{a_0}(m_2 - M_2)(a_2\cos 2\omega t + a_4\cos 4\omega t + \cdots),$$

$$\varepsilon_1 - 2\varepsilon_2 - 2\varepsilon_3 - \varepsilon_4 = \frac{2}{a_0}(m_3 - M_3)(a_2\cos 2\omega t + a_4\cos 4\omega t + \cdots),$$

$$\text{and} \qquad 3\varepsilon_1 - 2\varepsilon_2 - \varepsilon_3 = 0.$$

In deriving the last equation, the constant of integration is found to vanish.

Hence,

$$\begin{aligned} \varepsilon_1 &= \lambda(m_1 - M_1), \\ \varepsilon_2 &= \lambda(m_1 - M_1 - m_2 + M_2), \\ \varepsilon_3 &= \lambda(m_2 - M_2 - m_3 + M_3), \qquad (7) \\ \varepsilon_4 &= -\varepsilon_1, \end{aligned}$$

where $$\lambda = \frac{1}{a_0}(a_2\cos 2\omega t + a_4\cos 4\omega t + \cdots).$$

λ can be computed for any given B - H curve.

Consider now the fluxes through the three sectors of the ring of centre O_1.

Flux along $PQ = f(i - \varepsilon_1 + M_1)$,

" " $QR = f(i - \varepsilon_1 + \varepsilon_2 + M_2)$,

" " $RS = f(i - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + M_3)$.

Now, since magnetic induction is a solenoidal vector, i.e. lines of flux are conserved, it follows that a number of lines of flux emerge from the ring at P, Q and R. These are respectively,

$$f(i-\varepsilon_1+\varepsilon_2+\varepsilon_3+M_3) - f(i-\varepsilon_1+M_1),$$

$$f(i-\varepsilon_1+M_1) - f(i-\varepsilon_1+\varepsilon_2+M_2),$$

$$f(i-\varepsilon_1+\varepsilon_2+M_2) - f(i-\varepsilon_1+\varepsilon_2+\varepsilon_3+M_3),$$

or

$$(\varepsilon_2+\varepsilon_3+M_3-M_1)\, f'(i),$$

$$(-\varepsilon_2+M_1-M_2)\, f'(i)$$

$$(-\varepsilon_3+M_2-M_3)\, f'(i),$$

to a first approximation. It is assumed that these lines of induction leave the ring normally. This seems to be the only possibility consistent with the approximation made earlier (of considering only the average field strength along each of the sectors).

Hence, the field strength at O_1 in say the direction QO_1 is

$$\beta = -\frac{1}{a_0}\left\{\varphi'\left(\frac{E}{3\omega}\cos\omega t\right) - \frac{1}{2}a_0\right\}\left\{m_1 - m_2 - M_1 + M_2\right\} f'(i) + (M_1 - M_2)\, f'(i)$$

$$= (M_1 - M_2)\left\{\frac{1}{a_0} + \frac{1}{2} f'(i)\right\} - (m_1 - m_2)\left\{\frac{1}{a_0} - \frac{1}{2} f'(i)\right\}. \qquad (8)$$

Similar expressions can be written down for the remaining components α, γ of the field strength at O_1.

If the direction of the resultant field strength at O_1 is specified by ψ then

$$\tan\psi = \frac{\sqrt{3}\,(\beta-\gamma)}{\beta+\gamma-2\alpha} \qquad (9)$$

$$= \sqrt{3}\,\frac{m_2\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\} - M_2\left\{\frac{1}{a_0} + \frac{1}{2} b'(i)\right\}}{(m_3 - m_1)\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\} - (M_3 - M_1)\left\{\frac{1}{a_0} + \frac{1}{2} b'(i)\right\}}. \qquad (10)$$

When the magnets are in equilibrium, $\psi = \varphi$, and the equilibrium is undisturbed by switching off the current. When this is done $i = 0$ and (10) reduces to

$$\tan\psi = \sqrt{3}\,\frac{m_2}{m_3 - m_1}.$$

Hence,

$$\frac{m_2\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\} - M_2\left\{\frac{1}{a_0} + \frac{1}{2} b'(i)\right\}}{(m_3 - m_1)\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\} - (M_3 - M_1)\left\{\frac{1}{a_0} + \frac{1}{2} b'(i)\right\}} = \frac{m_2}{m_3 - m_1},$$

or

$$\frac{M_2}{M_3 - M_1} = \frac{m_2}{m_3 - m_1}.$$

Since $M_1 + M_2 + M_3 = m_1 + m_2 + m_3 = 0$, it follows immediately that

$$\frac{m_1}{m_2} = \frac{M_1}{M_2} \quad \text{and} \quad \frac{m_1}{m_3} = \frac{M_1}{M_3}. \qquad (11)$$

If the magnets are identical in shape, then by (11) they must be aligned parallel to each other. This can be shown more explicitly in the case when the magnets are dipoles. (2) and (10) with $\psi = \varphi$, give $\tan\varphi = \tan\theta$ or $\varphi = \theta$.

It should be observed from (8) that the field strength at O_1 consists of two parts. One of these, corresponding to terms such as $(m_1 - m_2)\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\}$ is directed along the axis of the magnet at O_1, while the other is in a direction parallel to the axis of the other magnet. For the purpose of finding the torque acting on either of the magnets, only one of these components need be considered. The magnitude of this component at O_1 is given by

$$H_2'^2 = -9\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\}^2 (m_1 m_3 + m_2 m_3 + m_1 m_2), \qquad (12)$$

and in the case when the magnets are dipoles, reduces to

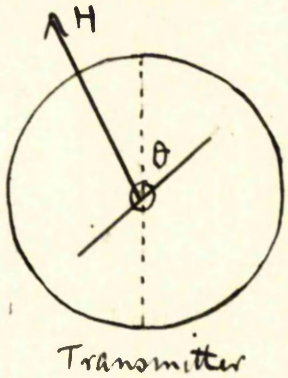
$$H_2' = \frac{27\mu}{4\pi a^3}\left\{\frac{1}{a_0} - \frac{1}{2} b'(i)\right\}.$$

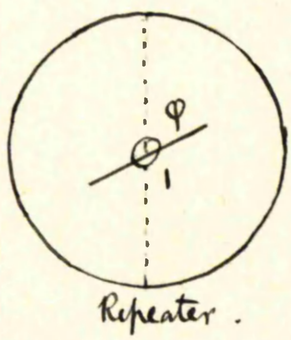
Similarly the corresponding field at O is

$$H_1' = \frac{27\mu'}{4\pi a^3}\left\{\frac{1}{a_0} - \frac{1}{2}\,b'(1)\right\}. \qquad (13)$$

In the remainder of this work, only the case of dipoles is considered. Further, the actual compass with the earth's field coming into play is contemplated now, rather than the idealised system (with no external field) considered above. It is convenient to regard the ring of centre O (which is unshielded) as the transmitter and the ring of centre O' (which is shielded) as the repeater.

When the repeater magnet is displaced from its equilibrium position (in which it is parallel to the transmitter and to the earth's field), it generates in the transmitter ring a magnetic field parallel to its new position. The transmitter magnet is thus acted on by the earth's field and a periodic field, and so assumes a direction intermediate between the directions of these fields. The magnetic field acting on the repeater magnet is also in this direction.





Further, it is the average values of the periodic fields (i.e. the constant term in the Fourier Expansion) which must be considered here; denote these constant parts by H_{1c}', H_{2c}' for/

for/ the transmitter and repeater respectively, cf(13).
Also, let the earth's field be denoted by H, and let its direction be specified by the angle ψ as in the diagram.

Thus $$\tan(\psi+\theta) = \frac{H'_{1c}\sin(\varphi+\psi)}{H + H'_{1c}\cos(\varphi+\psi)}, \qquad (14)$$

whence $$H'_{1c}\sin(\varphi-\theta) = H\sin(\theta+\psi). \qquad (15)$$

(14) determines the deflection of the transmitter magnet from its equilibrium position for given deflection of the repeater magnet.

Now, couple acting on repeater magnet

$$= \mu' H'_2 \sin(\varphi-\theta)$$
$$= \mu H'_1 \sin(\varphi-\theta).$$

Thus, average value of couple $= \mu H \sin(\theta+\psi)$ (16)
which is equal to the couple acting on the transmitter magnet when it is displaced through an angle $\theta+\psi$ from the direction of the earth's field. Thus action and reaction are equal and opposite as it were. (This I think is very satisfactory, but I do not think it would hold if the rings were of unequal radius).
Also, from (14), $$\sin(\theta+\psi) = \frac{H'_{1c}\sin(\varphi+\psi)}{\sqrt{[H^2 + H'^2_{1c} + 2HH'_{1c}\cos(\varphi+\psi)]}},$$
whence from (16)

Couple acting on repeater magnet $$= \frac{\mu H H'_1 \sin(\varphi+\psi)}{\sqrt{[H^2 + H'^2_{1c} + 2HH'_{1c}\cos(\varphi+\psi)]}}. \qquad (17)$$

(in deriving (17), the lines of force in the rings due to the earth's field have been neglected).

This expression for the couple is of exactly the kind found in practice, being zero for $\theta+\psi = 0^\circ, 180^\circ$, and in/

in/ general defining a skew curve. E' and H' can be computed for any given B - H curve, since by (3.1) $b'(i) = \{\varphi'(E\cos wt/3\omega)\}$. As a check on the reasonable nature of the theory (and these particular considerations do not require the rings to be of equal size), it is found that (17) gives maximum average couple when $\theta + \psi = 140^\circ$ (as in the actual compass) provided $H'_{1c}/H = 3/4$; from (14) it then follows that for small displacements $\psi + \theta = \frac{3}{7}(\psi + \varphi)$. In practice it is found that the one deflection is about half of the other, so that the agreement is satisfactory. Further, the torque per degree worked out on this basis is of the same order as that found in the actual compass. On this latter point, inequality in the sizes of the rings does make a difference, so complete agreement between experiment and our somewhat skeleton theory cannot be expected. (Differences in the windings would also have an effect).

As in the theory of the Fluxgate Magnetometer, the higher terms of certain Taylor expansions have been neglected. This is valid only if the field strengths due to the magnets are small enough to correspond to the straight part of the B - H curve. In practice, this is achieved by placing the magnets well above the toroids.

Conclusion

From the operation of the three fluxgate systems which we have considered, it would appear that the following rule is of general application.

If, in a system of alternating currents containing inductances and ferromagnetic materials, the currents are modified by the introduction of external magnetic fields whose strengths <u>in</u> the materials are small, then the change in the value of the current in any closed circuit containing iron, is proportional to the number of lines of force of the external fields passing through that circuit.

This enunciation of the rule is purely tentative and may require modifications both major and minor. It is of interest to see how it applies to the cases considered above. In the case of the Gyro-Fluxgate compass, the result is an elementary consequence of the rule. As for the Magnesyn system, reference to the diagram shows that

$$\delta i_1 = k m_1, \qquad \delta i_4 = l M_1,$$

$$\delta i_1 - \delta i_2 = k m_2, \qquad \delta i_2 + \delta i_4 = l M_2,$$

$$\delta i_1 - \delta i_2 - \delta i_3 = k m_3, \qquad \delta i_2 + \delta i_3 + \delta i_4 = l M_3,$$

where k, l are constants [Cf. eqns (7), (11)].

Since $m_1 + m_2 + m_3 = M_1 + M_2 + M_3 = 0$ it follows that

$$3\delta i_1 - 2\delta i_2 - \delta i_3 = 3\delta i_4 + 2\delta i_2 + \delta i_3 = 0,$$

whence

$$\delta i_4 = -\delta i_1 .$$

Thus $\dfrac{m_2}{m_1} = 1 - \dfrac{\delta i_2}{\delta i_1} = 1 + \dfrac{\delta i_2}{\delta i_4} = \dfrac{M_2}{M_1}$,

and similarly $\dfrac{m_2}{m_3} = \dfrac{M_2}{M_3}$.

As before, this means that if the magnets are identical in shape, then in the equilibrium state they are aligned parallel to each other.
